

Boundary Harnack principle and gradient estimates for fractional Laplacian perturbed by non-local operators

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Abstract

Suppose $d \geq 2$ and $0 < \beta < \alpha < 2$. We consider the non-local operator $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, where

$$\mathcal{S}^b f(x) := \lim_{\varepsilon \rightarrow 0} \mathcal{A}(d, -\beta) \int_{|z| > \varepsilon} (f(x+z) - f(x)) \frac{b(x, z)}{|z|^{d+\beta}} dy.$$

Here $b(x, z)$ is a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ that is symmetric in z , and $\mathcal{A}(d, -\beta)$ is a normalizing constant so that when $b(x, z) \equiv 1$, \mathcal{S}^b becomes the fractional Laplacian $\Delta^{\beta/2} := -(-\Delta)^{\beta/2}$. In other words,

$$\mathcal{L}^b f(x) := \lim_{\varepsilon \rightarrow 0} \mathcal{A}(d, -\beta) \int_{|z| > \varepsilon} (f(x+z) - f(x)) j^b(x, z) dz,$$

where $j^b(x, z) := \mathcal{A}(d, -\alpha)|z|^{-(d+\alpha)} + \mathcal{A}(d, -\beta)b(x, z)|z|^{-(d+\beta)}$. It is recently established in Chen and Wang [12] that, when $j^b(x, z) \geq 0$ on $\mathbb{R}^d \times \mathbb{R}^d$, there is a conservative Feller process X^b having \mathcal{L}^b as its infinitesimal generator. In this paper we establish, under certain conditions on b , a uniform boundary Harnack principle for harmonic functions of X^b (or equivalently, of \mathcal{L}^b) in any κ -fat open set. We further establish uniform gradient estimates for non-negative harmonic functions of X^b in open sets.

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1 Introduction

Let $d \geq 2$, $0 < \beta < \alpha < 2$, and $b(x, z)$ be a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ with $b(x, z) = b(x, -z)$ for $x, z \in \mathbb{R}^d$. Consider the non-local operator $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, where

$$\mathcal{S}^b f(x) := \lim_{\varepsilon \rightarrow 0} \mathcal{A}(d, -\beta) \int_{|z| > \varepsilon} (f(x+z) - f(x)) \frac{b(x, z)}{|z|^{d+\beta}} dz.$$

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Here $\mathcal{A}(d, -\beta)$ is a normalizing constant so that when $b(x, z) \equiv 1$, \mathcal{S}^b becomes the fractional Laplacian $\Delta^{\beta/2} := -(-\Delta)^{\beta/2}$; in other words, $\mathcal{A}(d, -\beta) = \beta 2^{\beta-1} \pi^{-d/2} \Gamma((d + \beta)/2) / \Gamma(1 - \beta/2)$. Thus \mathcal{L}^b can be expressed as

$$\mathcal{L}^b f(x) = \lim_{\varepsilon \rightarrow 0} \int_{|z| > \varepsilon} (f(x + z) - f(x)) j^b(x, z) dz$$

where

$$j^b(x, z) = \frac{\mathcal{A}(d, -\alpha)}{|z|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta)b(x, z)}{|z|^{d+\beta}}.$$

Note that since $b(x, z)$ is symmetric in z , for $f \in C_b^2(\mathbb{R}^d)$,

$$\mathcal{L}^b f(x) = \int_{\mathbb{R}^d} (f(x + z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}}) j^b(x, z) dz.$$

Recently, \mathcal{L}^b , the fractional Laplacian perturbed by a lower order non-local operator \mathcal{S}^b , and its fundamental solution have been studied in Chen and Wang [12]. It is established there that if for every $x \in \mathbb{R}^d$, $j^b(x, z) \geq 0$ (that is, $b(x, z) \geq -\frac{\mathcal{A}(d, -\alpha)}{\mathcal{A}(d, -\beta)} |z|^{\beta-\alpha}$ for a.e. $z \in \mathbb{R}^d$), then \mathcal{L}^b has a unique jointly continuous fundamental solution $p^b(t, x, y)$, which uniquely determines a conservative Feller process X^b on the canonical Skorokhod space $\mathbb{D}([0, +\infty), \mathbb{R}^d)$ such that

$$\mathbb{E}_x [f(X_t^b)] = \int_{\mathbb{R}^d} f(y) p^b(t, x, y) dy, \quad x \in \mathbb{R}^d.$$

for every bounded measurable function f on \mathbb{R}^d . The Feller process X^b is typically non-symmetric and it has a Lévy system $(J^b(x, y) dy, t)$ (see [12, Theorem 1.3]), where

$$J^b(x, y) := j^b(x, y - x). \quad (1.1)$$

When b takes constant value $\varepsilon > 0$, X^ε has the same distribution as the Lévy process $Y + \varepsilon^{1/\beta} Z$, where Y and Z are rotationally symmetric α - and β -stable processes on \mathbb{R}^d that are independent of each other. Moreover, two-sided heat kernel estimates for \mathcal{L}^b have been obtained in Chen and Wang [12], while two-sided Dirichlet kernel estimates in $C^{1,1}$ open sets have recently been obtained in Chen and Yang [13]. We emphasize that the lower order perturbations \mathcal{S}^b considered in [12, 13] and in this paper are of high intensity, in the sense that $\int_{\mathbb{R}^d} \frac{|b(x, z)|}{|z|^{d+\beta}} dz$ may be identically infinite. This is the case, for example, when $b(x, z)$ is bounded away from 0. Such high intensity non-local perturbation is in stark contrast to the Feynman-Kac type non-local perturbation considered in [5, 9, 11, 15, 16], where the kernels used in the non-local perturbation are typically integrable. Under high intensity non-local perturbation, the potential theory for the perturbed generator \mathcal{L}^b can be different from that of $\Delta^{\alpha/2}$. For example, the heat kernel of \mathcal{L}^b may not be comparable to that of $\Delta^{\alpha/2}$; see [12] for details.

In this paper, we investigate boundary Harnack principle and gradient estimates for non-negative harmonic functions of \mathcal{L}^b in open sets. Boundary Harnack principle (BHP) asserts that non-negative harmonic functions that vanish in an exterior part of a neighborhood at the boundary decay at the same rate. It is an important property in analysis and in probability theory on harmonic functions. We refer the reader to the introduction of [3, 17] for a brief account on the history

of BHP that started with Brownian motion and then extended to subordinate Brownian motions and to certain pure jump strong Markov processes. Since \mathcal{L}^b is typically state-dependent and its dual operator may not be Markovian, the BHP results in [3, 17] are not applicable to harmonic functions of \mathcal{L}^b . In this paper, we establish uniform boundary Harnack principle on κ -fat open sets for non-negative harmonic functions of \mathcal{L}^b by estimating Poisson kernels of \mathcal{L}^b in small balls.

Gradient estimates for harmonic functions of elliptic operators and on manifolds have been studied extensively in literature, including the celebrated Li-Yau inequality. See [14] and the references therein and for a coupling argument. See also [21] for gradient estimates of the transition semigroups of some Lévy processes using coupling method. Gradient estimates for harmonic functions for non-local operators are quite recently. In [4], a gradient estimate for harmonic functions of symmetric stable processes is obtained. Gradient estimates for harmonic functions of mixed stable processes were derived in [22]. It has recently been extended to a class of isotropic unimodal Lévy process in [20]. For gradient estimate for harmonic functions of the Schrödinger operator $\Delta^{\alpha/2} + q$, see [4] for $\alpha \in (1, 2)$ and [19] for $\alpha \in (0, 1]$. The second main result of this paper is to establish gradient estimates for positive harmonic functions of \mathcal{L}^b . As far as we know, this is the first gradient estimate result for non-Lévy non-local operators.

We now describe our main results in details. In this paper, we use “:=” as a way of definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Let $|x - y|$ denote the Euclidean distance between x and y , and $B(x, r)$ the open ball centered at x with radius $r > 0$. For any two positive functions f and g , $f \stackrel{c}{\lesssim} g$ means that there is a positive constant c such that $f \leq cg$ on their common domain of definition, and $f \stackrel{c}{\asymp} g$ means that $c^{-1}g \leq f \leq cg$. We also write “ \lesssim ” and “ \asymp ” if c is unimportant or understood. If $D \subset \mathbb{R}^d$ is an open set, for every $x, y \in D$, define

$$\delta_D(x) := \text{dist}(x, \partial D) \quad \text{and} \quad r_D(x, y) := \delta_D(x) + \delta_D(y) + |x - y|. \quad (1.2)$$

It is easy to see that

$$r_D(x, y) \asymp \delta_D(x) + |x - y| \asymp \delta_D(y) + |x - y|. \quad (1.3)$$

Denote by $\tau_D^b := \inf\{t > 0 : X_t^b \notin D\}$, the exit time from D by X^b . When there is no danger of confusion, we will drop the superscript b and simply write τ_D for τ_D^b .

Definition 1.1. A function f defined on \mathbb{R}^d is said to be harmonic in an open set D with respect to X^b if it has the mean-value property: for every bounded open set $U \subset D$ with $\overline{U} \subset D$,

$$f(x) = \mathbb{E}_x \left[f(X_{\tau_U}^b) \right] \quad \text{for } x \in U. \quad (1.4)$$

It is said to be regular harmonic in D if (1.4) holds for $U = D$.

Denote by ∂ a cemetery point that is added to D as an isolated point. We use the convention that $X_\infty^b := \partial$ and any function f is extended to the cemetery point ∂ by setting $f(\partial) = 0$. So $\mathbb{E}_x [f(X_{\tau_D}^b)]$ should be understood as $\mathbb{E}_x [f(X_{\tau_D}^b) : \tau_D < +\infty]$. In Definition 1.1, we always assume implicitly that the expectation in (1.4) is absolutely convergent.

Assumption 1 Suppose $M_1, M_2 \geq 1$. $b(x, z)$ is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$\|b\|_\infty \leq M_1 \quad \text{and} \quad b(x, z) = b(x, -z) \quad \text{for } x, z \in \mathbb{R}^d, \quad (1.5)$$

and there exists a positive constant $\varepsilon_0 \in [0, 1]$ such that for every $x, y \in \mathbb{R}^d$,

$$M_2^{-1} J^{\varepsilon_0}(x, y) \leq J^b(x, y) \leq M_2 J^{\varepsilon_0}(x, y). \quad (1.6)$$

Here $J^{\varepsilon_0}(x, y) = \mathcal{A}(d, -\alpha)|x - y|^{-d-\alpha} + \varepsilon_0 \mathcal{A}(d, -\beta)|x - y|^{-d-\beta}$. Since $J^{\varepsilon_0}(x, y)$ depends only on $|x - y|$, we also write $J^{\varepsilon_0}(|x - y|)$ for $J^{\varepsilon_0}(x, y)$.

Definition 1.2. Let $\kappa \in (0, 1)$. An open set $D \subset \mathbb{R}^d$ is said to be κ -fat if for every $z \in \partial D$ and $r \in (0, 1]$, there is some point $x \in D$ so that $B(x, \kappa r) \subset D \cap B(z, r)$.

The following is the first main result of this paper.

Theorem 1.3 (Uniform boundary Harnack inequality). *Suppose Assumption 1 holds and D is a κ -fat open set in \mathbb{R}^d with $\kappa \in (0, 1)$. There exist constants $r_1 = r_1(d, \alpha, \beta, M_1) \in (0, 1]$ and $C_1 = C_1(d, \alpha, \beta, \kappa, M_1, M_2) \geq 1$ such that for every $z_0 \in \partial D$ and $r \in (0, r_1/2]$, and all non-negative functions u, v that are regular harmonic in $D \cap B(z_0, 2r)$ with respect to X^b and vanish in $D^c \cap B(z_0, 2r)$, we have*

$$\frac{u(x)}{v(x)} \leq C_1 \frac{u(y)}{v(y)} \quad \text{for } x, y \in D \cap B(z_0, r).$$

We call the above property *uniform boundary Harnack principle* because the constants r_1 and C_1 in the above theorem are independent of $\varepsilon_0 \in [0, 1]$ appeared in condition (1.6). We next study the gradient estimates for non-negative harmonic functions in open sets. We write ∂_{x_i} or ∂_i for $\frac{\partial}{\partial x_i}$ and ∇ for $(\partial_{x_1}, \dots, \partial_{x_d})$.

Theorem 1.4. *Let D be an arbitrary open set in \mathbb{R}^d . Under Assumption 1, there is a constant $C_2 = C_2(d, \alpha, \beta, M_1, M_2) > 0$ such that for any non-negative function f in \mathbb{R}^d which is harmonic in D with respect to X^b , $\nabla f(x)$ exists for every $x \in D$, and we have*

$$|\nabla f(x)| \leq C_2 \frac{f(x)}{1 \wedge \delta_D(x)} \quad \text{for } x \in D.$$

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $1 \leq i \leq d$, we write \tilde{x}^i for $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \in \mathbb{R}^{d-1}$.

Assumption 2. Suppose there is $i \in \{1, \dots, d\}$ so that for every $x \in \mathbb{R}^d$,

$$b(x, z) = \varphi(\tilde{x}^i) \psi(|z|) \quad \text{a.e. } z \in \mathbb{R}^d,$$

where $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a non-negative measurable function, and $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a measurable function such that

$$\frac{\psi(r)}{r^{d+\beta}} \quad \text{is non-increasing in } r > 0. \quad (1.7)$$

Theorem 1.5. Suppose Assumption 1 and Assumption 2 hold. Let $D = \{x \in \mathbb{R}^d : x_i > \Gamma(\tilde{x}^i)\}$ be a special Lipschitz domain, where $\Gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is a Lipschitz function with Lipschitz constant λ_0 (that is, $|\Gamma(\tilde{x}^i) - \Gamma(\tilde{y}^i)| \leq \lambda_0 |\tilde{x}^i - \tilde{y}^i|$ for every $\tilde{x}^i, \tilde{y}^i \in \mathbb{R}^{d-1}$). Then there are positive constants $R_1 = R_1(d, \alpha, \beta, \lambda_0, M_1, M_2)$ and $C_3 = C_3(d, \alpha, \beta, \lambda_0, M_1, M_2) \geq 1$ such that for every $r \in (0, R_1]$, there is a constant $\eta_1 = \eta_1(d, \alpha, \beta, \lambda_0, M_1, M_2, r) \in (0, r/2)$ so that for every $z_0 \in \partial D$ and every non-negative function f that is harmonic in $D \cap B(z_0, r)$ with respect to X^b and vanishes in $D^c \cap B(z_0, r)$,

$$C_3^{-1} \frac{f(x)}{\delta_D(x)} \leq |\nabla f(x)| \leq C_3 \frac{f(x)}{\delta_D(x)} \quad \text{for } x \in D \cap B(z_0, \eta_1). \quad (1.8)$$

Obviously Assumption 2 is implied by

Assumption 3. There exists a measurable function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying (1.7) such that for every $x \in \mathbb{R}^d$,

$$b(x, z) = \psi(|z|) \quad \text{a.e. } z \in \mathbb{R}^d.$$

Definition 1.6. An open set $D \subset \mathbb{R}^d$ is said to be Lipschitz if for every $z_0 \in \partial D$, there is a Lipschitz function $\Gamma_{z_0} : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, an orthonormal coordinate system CS_{z_0} and a constant $R_{z_0} > 0$ such that if $y = (y_1, \dots, y_{d-1}, y_d)$ in CS_{z_0} coordinates, then

$$D \cap B(z_0, R_{z_0}) = \{y : y_d > \Gamma_{z_0}(y_1, \dots, y_{d-1})\} \cap B(z_0, R_{z_0}).$$

If there exist positive constants R_0 and λ_0 so that R_{z_0} can be taken to be R_0 for all $z_0 \in \partial D$ and the Lipschitz constants of Γ_{z_0} are not greater than λ_0 , we call D a Lipschitz open set with characteristics (λ_0, R_0) .

Clearly, if D is a Lipschitz open set with characteristics (λ_0, R_0) , then it is κ -fat for some $\kappa = \kappa(\lambda_0, R_0) \in (0, 1)$. The following theorem follows directly from Theorem 1.5.

Theorem 1.7. Let D be a Lipschitz open set in \mathbb{R}^d with characteristics (λ_0, R_0) . Under Assumptions 1 and 3, there are positive constants $R_2 = R_2(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2)$ and $C_4 = C_4(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2) \geq 1$ such that for every $r \in (0, R_2]$, there is a constant $\eta_2 = \eta_2(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2, r) \in (0, r/2)$ so that for every $z_0 \in \partial D$ and every non-negative function f that is harmonic in $D \cap B(z_0, r)$ with respect to X^b and vanishes in $D^c \cap B(z_0, r)$,

$$C_4^{-1} \frac{f(x)}{\delta_D(x)} \leq |\nabla f(x)| \leq C_4 \frac{f(x)}{\delta_D(x)} \quad \text{for } x \in D \cap B(z_0, \eta_2).$$

Results in Theorem 1.4, Theorem 1.5 and Theorem 1.7 can be called *uniform* gradient estimates because the constants C_k , $2 \leq k \leq 4$, and η_i , $1 \leq i \leq 2$, are independent of ε_0 of (1.6).

The rest of the paper is organized as follows. Preliminary results on Green functions and Poisson kernels are presented in Section 2. The proof of the uniform boundary Harnack principle is given in Section 3. Section 4 is devoted to the proof of Theorem 1.4, while the proof of Theorem 1.5 is given in Section 5. In this paper, we use capital letters C_1, C_2, \dots to denote constants in the statements of results. The lower case constants c_1, c_2, \dots , will denote the generic constants used in proofs, whose exact values are not important, and can change from one appearance to another. We use e_k to denote the unit vector along the positive direct of x_k -axis.

2 Preliminaries

Recall the Lévy system $(J^b(x, y)dy, t)$ from (1.1), which describes the jumps of X^b : for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, $x \in \mathbb{R}^d$ and stopping time T (with respect to the filtration of X^b),

$$\mathbb{E}_x \left[\sum_{s \leq T} f(s, X_{s-}^b, X_s^b) \right] = \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}^d} f(s, X_s^b, y) J^b(X_s^b, y) dy ds \right]. \quad (2.1)$$

Suppose D is a Greenian open set of X^b . Let $G_D^b(x, y)$ denote the Green function of D , that is,

$$\int_D f(y) G_D^b(x, y) dy = \mathbb{E}_x \left[\int_0^{\tau_D} f(X_s^b) ds \right]$$

for every bounded measurable function f on D and $x \in D$. It follows from (2.1) that for every bounded open set D in \mathbb{R}^d , every $f \geq 0$, and $x \in D$,

$$\mathbb{E}_x \left[f(X_{\tau_D}^b) : X_{\tau_D-}^b \neq X_{\tau_D}^b \right] = \int_{\bar{D}^c} f(z) \left(\int_D G_D^b(x, y) J^b(y, z) dy \right) dz. \quad (2.2)$$

Define

$$K_D^b(x, z) := \int_D G_D^b(x, y) J^b(y, z) dy \quad \text{for } (x, z) \in D \times \bar{D}^c. \quad (2.3)$$

We call $K_D^b(x, z)$ the Poisson kernel of X^b on D . Then (2.2) can be written as

$$\mathbb{E}_x \left[f(X_{\tau_D}^b) : X_{\tau_D-}^b \neq X_{\tau_D}^b \right] = \int_{\bar{D}^c} f(z) K_D^b(x, z) dz.$$

For any $\lambda > 0$, define

$$b_\lambda(x, z) := \lambda^{\beta-\alpha} b(\lambda^{-1}x, \lambda^{-1}z) \quad \text{for } x, z \in \mathbb{R}^d.$$

It is not hard to show that

$$J^{b_\lambda}(x, y) = \lambda^{-(d+\alpha)} J^b(\lambda^{-1}x, \lambda^{-1}y) \quad \text{for } x, y \in \mathbb{R}^d. \quad (2.4)$$

and

$$\{\lambda X_{\lambda^{-1}t}^b; t \geq 0\} \text{ has the same distribution as } \{X_t^{b_\lambda}; t \geq 0\}.$$

So for any $\lambda > 0$, we have the following scaling properties:

$$G_D^b(x, y) = \lambda^{d-\alpha} G_{\lambda D}^{b_\lambda}(\lambda x, \lambda y) \quad \text{for } x, y \in D, \quad (2.5)$$

$$K_D^b(x, z) = \lambda^d K_{\lambda D}^{b_\lambda}(\lambda x, \lambda z) \quad \text{for } x \in D, z \in \bar{D}^c. \quad (2.6)$$

If u is harmonic in D with respect to X^b , then for any $\lambda > 0$, $v(x) := u(x/\lambda)$ is harmonic in λD with respect to X^{b_λ} .

When $b(x, z) \equiv 0$, X^0 is simply an isotropic symmetric α -stable process on \mathbb{R}^d , which we will denote as X . We will also write J for J^0 . It is known that if $d > \alpha$, the process X is transient and its Green function is given by

$$G(x, y) = \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2} |x - y|^{\alpha-d} \quad \text{for } x, y \in \mathbb{R}^d. \quad (2.7)$$

It is shown in Blumenthal *et al.* [1] that the Green function of X in a ball $B(0, r)$ is given by

$$G_{B(0,r)}(x, y) = \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2} \int_0^z (u+1)^{-d/2} u^{\alpha/2-1} du |x-y|^{\alpha-d} \quad \text{for } x, y \in B(0, r), \quad (2.8)$$

where $z = (r^2 - |x|^2)(r^2 - |y|^2)|x-y|^{-2}$ and $r > 0$. The above formula yields the following two-sided estimates (see, for example, [6]): Suppose B is an arbitrary ball in \mathbb{R}^d with radius $r > 0$. Then there is a universal constant $c_1 = c_1(d, \alpha) > 1$ so that for every $x, y \in B$,

$$G_B(x, y) \asymp_{c_1} |x-y|^{\alpha-d} \left(1 \wedge \frac{\delta_B(x)}{|x-y|}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_B(y)}{|x-y|}\right)^{\alpha/2}. \quad (2.9)$$

Since for $a, b > 0$, $a \wedge b \asymp \frac{ab}{a+b}$ and $1 \wedge \frac{a}{b} \asymp \frac{a}{a+b}$, in view of (1.3) we can rewrite (2.9) as

$$G_B(x, y) \asymp |x-y|^{\alpha-d} \frac{\delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2}}{r_B(x, y)^\alpha}. \quad (2.10)$$

It follows immediately from (2.9) that there is a positive constant $c_2 = c_2(d, \alpha) > 1$ so that for $B = B(x_0, r)$,

$$c_2^{-1} r^\alpha \leq \mathbb{E}_x \tau_B \leq c_2 r^\alpha \quad \text{for } x \in B(x_0, r/2).$$

Riesz (see [1]) derived the following explicit formula for the Poisson kernel $K_{B(0,r)}(x, z)$ of X on $B(0, r)$.

$$K_{B(0,r)}(x, z) = \frac{\Gamma(d/2) \sin(\pi\alpha/2)}{\pi^{d/2+1}} \frac{(r^2 - |x|^2)^{\alpha/2}}{(|z|^2 - r^2)^{\alpha/2} |x-z|^d} \quad \text{for } |x| < r \text{ and } |z| > r, \quad (2.11)$$

We also know from [2, Lemma 6] that, for rotationally symmetric α -stable process X ,

$$\mathbb{P}_x(X_{\tau_D} \in \partial D) = 0 \quad \text{and so} \quad \mathbb{P}_x(X_{\tau_D} \neq X_{\tau_D-}) = 1$$

for every $x \in D$ if D is a domain that satisfies uniform exterior cone condition. (Although the result of [2, Lemma 6] is stated only for bounded domain satisfying uniform exterior cone condition, its proof works for any domain satisfying uniform exterior cone condition.)

3 Boundary Harnack principle

Recall that we write X and J for X^0 and J^0 , respectively. First we record the following gradient estimate on the Green function G_D of symmetric α -stable process X from [4].

Lemma 3.1. ([4, Corollary 3.3]) *Let D be a Greenian domain in \mathbb{R}^d of X . Then*

$$|\nabla G_D(x, y)| \leq d \frac{G_D(x, y)}{|x-y| \wedge \delta_D(x)} \quad \text{for } x, y \in D, \ x \neq y. \quad (3.1)$$

For $x \neq y$ in D , define

$$h_D(x, y) := \begin{cases} |x - y|^{\alpha - \beta - d} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2} & \text{if } \alpha > 2\beta, \\ |x - y|^{\beta - d} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\beta} \left(1 \vee \log \frac{|x - y|}{\delta_D(x)}\right) & \text{if } \alpha = 2\beta, \\ |x - y|^{\alpha - \beta - d} \left(1 \wedge \frac{\delta_D(y)}{|x - y|}\right)^{\alpha/2} \left(1 \vee \frac{|x - y|}{\delta_D(x)}\right)^{\beta - \alpha/2} & \text{if } \alpha < 2\beta, \end{cases} \quad (3.2)$$

The following two results are established in [13].

Lemma 3.2. ([13, Theorems 4.11 and Lemma 4.13]) *Suppose b is a bounded function satisfying (1.5), and that for every $x \in \mathbb{R}^d$, $J^b(x, y) \geq 0$ a.e. $y \in \mathbb{R}^d$. There exist positive constants $r_1 = r_1(d, \alpha, \beta, M_1) \in (0, 1]$ and $C_5 = C_5(d, \alpha, \beta, M_1)$ such that for any $x_0 \in \mathbb{R}^d$ and any ball $B = B(x_0, r)$ with radius $r \in (0, r_1]$, we have for $x, y \in B$,*

$$\frac{1}{2}G_B(x, y) \leq G_B^b(x, y) \leq \frac{3}{2}G_B(x, y) \quad \text{and} \quad |S_x^b G_B^b(x, y)| \leq C_5 h_B(x, y) \quad (3.3)$$

Moreover, $\mathbb{P}_x(X_{\tau_B}^b \in \partial B) = 0$ for every $x \in B$. In this case, for every non-negative measurable function f ,

$$\mathbb{E}_x f(X_{\tau_B}^b) = \int_{\bar{B}^c} f(z) K_B^b(x, z) dz \quad \text{for } x \in B.$$

Lemma 3.3. ([13, Lemma 4.2]) *Let D be a bounded open set in \mathbb{R}^d . There exists a constant $C_6 = C_6(d, \alpha, \beta, \text{diam}(D), M_1) > 0$ such that for any bounded function b satisfying (1.5), and that for every $x \in \mathbb{R}^d$, $J^b(x, y) \geq 0$ for a.e. $y \in \mathbb{R}^d$, we have*

$$G_D^b(x, y) \leq C_6 |x - y|^{\alpha - d} \quad \text{for } x, y \in D.$$

Note that the constant C_7 below is independent of $\varepsilon_0 \in [0, 1]$ appeared in (1.6).

Theorem 3.4 (Uniform Harnack inequality). *Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. Under Assumption 1, there exists a constant $C_7 = C_7(d, \alpha, \beta, M_1, M_2) \geq 1$ such that for every $x_0 \in \mathbb{R}^d$, $r \in (0, r_1]$, and every non-negative function u which is regular harmonic in $B(x_0, r)$, we have*

$$\sup_{y \in B(x_0, r/2)} u(y) \leq C_7 \inf_{y \in B(x_0, r/2)} u(y).$$

Proof. Let $u^*(x) := \mathbb{E}_x \left[u(X_{\tau_{B(x_0, r)}}^{\varepsilon_0}) \right]$. Then u^* is regular harmonic in $B(x_0, r)$ with respect to the mixed stable processes X^{ε_0} . In view of Lemma 3.2 and Assumption 1, for every $x_0 \in \mathbb{R}^d$ and $r \in (0, r_1]$, the Poisson kernel $K_{B(x_0, r)}^b(x, z)$ on $B(x_0, r)$ of X^b is comparable to that of X^{ε_0} . Thus for every $x \in B(x_0, r/2)$, $u(x)$ is comparable to $u^*(x)$. Theorem 3.4 then follows from the uniform Harnack inequality for mixed stable processes; see [7, (3.40)]. \square

Lemma 3.5 (Harnack inequality). *Under Assumption 1, there exists a constant $C_8 = C_8(d, \alpha, \beta, M_1, M_2) > 0$ such that the following statement is true: If $x_1, x_2 \in \mathbb{R}^d$, $r \in (0, r_1]$ and $k \in \mathbb{N}$ are such that $|x_1 - x_2| < 2^k r$, then for every non-negative function u which is harmonic with respect to X^b in $B(x_1, r) \cup B(x_2, r)$, we have*

$$C_8^{-1} 2^{-k(d+\alpha)} u(x_2) \leq u(x_1) \leq C_8 2^{k(d+\alpha)} u(x_2). \quad (3.4)$$

Proof. Without loss of generality, we may assume $|x_1 - x_2| \geq r/4$. Note that for every $x \in B(x_2, r/8) \subset B(x_1, r/8)^c$, we have $|x - x_1| < 2^{k+1}r$. Thus by Lemma 3.2 and Assumption 1, we have

$$\begin{aligned} K_{B(x_1, r/8)}^b(x_1, x) &\geq \frac{1}{2M_2} \int_{B(x_1, r/8)} G_{B(x_1, r/8)}(x_1, y) J(y, x) dy \\ &= \frac{1}{2M_2} K_{B(x_1, r/8)}(x_1, x) \\ &\geq \frac{c_1}{2M_2} 2^{-\alpha} r^\alpha (2^{-k-1}r)^{-d-\alpha} = c_2 r^{-d} 2^{-k(d+\alpha)}. \end{aligned} \quad (3.5)$$

Recall that by Theorem 3.4, we have $u(x) \geq c_3 u(x_2)$ for every $x \in B(x_2, r/8)$. Thus by (3.5),

$$\begin{aligned} u(x_1) &\geq \int_{B(x_2, r/8)} u(x) K_{B(x_1, r/8)}^b(x_1, x) dx \\ &\geq c_2 c_3 u(x_2) r^{-d} 2^{-k(d+\alpha)} \int_{B(x_2, r/8)} dx \\ &\geq c_4 2^{-k(d+\alpha)} u(x_2), \end{aligned}$$

and (3.4) follows by symmetry. \square

Proof of Theorem 1.3. Note that there are constants $R_0 = R_0(d, \alpha, \beta, M_1) \in (0, r_1)$ and $c = c(d, \alpha, \beta, M_1) > 1$ so that

$$\frac{c^{-1}}{|x - y|^{d+\alpha}} \leq J^b(x, y) \leq \frac{c}{|x - y|^{d+\alpha}}$$

for all $|x - y| \leq R_0$ and $b(x, z)$ satisfying (1.5). Thus using (3.3), we can get uniform estimates on the Poisson kernel

$$K_{B(x_0, r)}^b(x, z) = \int_{B(x_0, r)} G_{B(x_0, r)}^b(x, y) J^b(y, z) dy$$

of any ball $B(x_0, r)$ with respect to X^b with $r \in (0, R_0/3)$, $x \in B(x_0, r)$ and $r < |z - x_0| < 2r$. Specifically, for $r < |z - x_0| < 2r$, $K_{B(x_0, r)}^b(x, z)$ is uniformly comparable to $K_{B(x_0, r)}(x, z)$. Using the explicit formula (2.11) for the Poisson kernel $K_{B(x_0, r)}$, (3.3), Theorem 3.4 and (1.6), we can adapt the arguments in [8, Theorem 2.6] to get our uniform boundary Harnack principle 1.3 (cf. the proof of [7, Theorem 3.9]). Since the proof is almost identical to those in [8, Section 3], we omit the details here. \square

Lemma 3.6. *Suppose Assumption 1 holds and D is a Lipschitz open set with characteristics (λ_0, R_0) . Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. There is a positive constant $C_9 = C_9(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2) \geq 1$ such that for every $z_0 \in \partial D$, $r \in (0, r_1/2)$, and every non-negative harmonic function u that is regular harmonic in $D \cap B(z_0, 2r)$ with respect to X^b and vanishes in $D^c \cap B(z_0, 2r)$,*

$$\mathbb{E}_x \left[u(X_{\tau_{D \cap B_k}}^b) : X_{\tau_{D \cap B_k}}^b \in B_0^c \right] \leq C_9 2^{-k\alpha} u(x), \quad x \in D \cap B_k, \quad (3.6)$$

for $B_k := B(z_0, 2^{-k}r)$ and $k \geq 1$.

Proof. Without loss of generality, we may assume $z_0 = 0$. By the uniform inner cone property of a Lipschitz open set, one can find a point $\tilde{z}_0 \in D \cap B(0, r)$ and $\kappa = \kappa(\lambda_0, R_0) \in (0, 1)$ such that $\tilde{B}_k := B(\tilde{z}_k, \kappa 2^{-k} r) \subset B_k \cap D$ for every $\tilde{z}_k := 2^{-k} \tilde{z}_0$ and $k \geq 0$. Define

$$u_k(x) := \mathbb{E}_x \left[u(X_{\tau_{D \cap B_k}}^b) : X_{\tau_{D \cap B_k}}^b \in B_0^c \right].$$

Since $u_0 = u$, (3.6) is clearly true for $k = 0$. Henceforth we assume $k \geq 1$. Note that $u_k \geq 0$ is regular harmonic with respect to X^b in $D \cap B_k$, and $u_k(x) \leq u_{k-1}(x)$ for all $x \in \mathbb{R}^d$. Define

$$I_k(x) := \mathbb{E}_x \left[u(X_{\tau_{B_k}}^b) : X_{\tau_{B_k}}^b \in B_0^c \right].$$

Clearly by definition $u_k(\tilde{z}_k) \leq I_k(\tilde{z}_k)$. For any $k \geq 1$, by Lemma 3.2 and (2.5), we have

$$\begin{aligned} K_{B_k}^b(\tilde{z}_k, y) &= \int_{B_k} G_{B_k}^b(\tilde{z}_k, z) J^b(z, y) dz \\ &\leq \frac{3}{2} \int_{B_k} G_{B_k}(\tilde{z}_k, z) J^b(z, y) dz \\ &= \frac{3}{2} \int_{2^{-(k-1)} B_1} G_{2^{-(k-1)} B_1}(2^{-(k-1)} \tilde{z}_1, z) J^b(z, y) dz \\ &= \frac{3}{2} 2^{-(k-1)d} \int_{B_1} G_{2^{-(k-1)} B_1}(2^{-(k-1)} \tilde{z}_1, 2^{-(k-1)} w) J^b(2^{-(k-1)} w, y) dw \\ &= \frac{3}{2} 2^{-(k-1)\alpha} \int_{B_1} G_{B_1}(\tilde{z}_1, w) J^b(2^{-(k-1)} w, y) dw. \end{aligned} \tag{3.7}$$

Note that for any $y \in B_0^c$ and $w \in B_1$,

$$\frac{|y - w|}{|y - 2^{-(k-1)} w|} \leq \frac{|y| + |w|}{|y| - 2^{-k+1}|w|} \leq 3.$$

Thus by (1.6) we have

$$J^b(2^{-(k-1)} w, y) \leq M_2 J^{\varepsilon_0}(|y - 2^{-(k-1)} w|) \leq 3^{d+\alpha} M_2 J^{\varepsilon_0}(|y - w|) \leq 3^{d+\alpha} M_2^2 J^b(w, y). \tag{3.8}$$

It follows from (3.7), (3.8) and Lemma 3.2 that for any $y \in B_0^c$,

$$\begin{aligned} K_{B_k}^b(\tilde{z}_k, y) &\leq \frac{3^{d+\alpha+1}}{2} M_2^2 2^{-(k-1)\alpha} \int_{B_1} G_{B_1}(\tilde{z}_1, w) J^b(w, y) dw \\ &\leq 3^{d+\alpha+1} M_2^2 2^{-(k-1)\alpha} \int_{B_1} G_{B_1}^b(\tilde{z}_1, w) J^b(w, y) dw \\ &= c_1 2^{-k\alpha} K_{B_1}^b(\tilde{z}_1, y). \end{aligned}$$

Now we have for $k \geq 1$

$$\begin{aligned} I_k(\tilde{z}_k) &= \int_{B_0^c} u(y) K_{B_k}^b(\tilde{z}_k, y) dy \\ &\leq c_1 2^{-k\alpha} \int_{B_0^c} u(y) K_{B_1}^b(\tilde{z}_1, y) dy \end{aligned}$$

$$= c_1 2^{-k\alpha} I_1(\tilde{z}_1). \quad (3.9)$$

Next we compare $I_1(\tilde{z}_1)$ with $u(\tilde{z}_1)$. Using Lemma 3.2, (1.6) and (2.5), we have

$$\begin{aligned} K_{\tilde{B}_1}^b(\tilde{z}_1, y) &= \int_{|z-\tilde{z}_1|<\kappa r/2} G_{\tilde{B}_1}^b(\tilde{z}_1, z) J^b(z, y) dz \\ &\geq \frac{1}{2M_2} \int_{|z-\tilde{z}_1|<\kappa r/2} G_{\tilde{B}_1}(\tilde{z}_1, z) J^{\varepsilon_0}(|y-z|) dz \\ &= \frac{1}{2M_2} \int_{|z-\tilde{z}_1|<\kappa r/2} G_{\kappa B_1}(0, z-\tilde{z}_1) J^{\varepsilon_0}(|y-z|) dz \\ &= \frac{1}{2M_2} \int_{|w|<\kappa r/2} G_{\kappa B_1}(0, w) J^{\varepsilon_0}(|y-\tilde{z}_1-w|) dw \\ &= \frac{1}{2M_2} \kappa^d \int_{|z|<r/2} G_{\kappa B_1}(0, \kappa z) J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa z|) dz \\ &= \frac{1}{2M_2} \kappa^\alpha \int_{B_1} G_{B_1}(0, z) J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa z|) dz. \end{aligned} \quad (3.10)$$

Again using Lemma 3.2 and (1.6), we have

$$\begin{aligned} K_{B_1}^b(\tilde{z}_1, y) &= \int_{B_1} G_{B_1}^b(\tilde{z}_1, z) J^b(z, y) dz \\ &\leq \frac{3}{2} M_2 \int_{B_1} G_{B_1}(\tilde{z}_1, z) J^{\varepsilon_0}(|y-z|) dz \\ &= \frac{3}{2} M_2 \left(\int_{|z|\leq|\tilde{z}_1|/2} + \int_{|\tilde{z}_1|/2<|z|<r/2} G_{B_1}(\tilde{z}_1, z) J^{\varepsilon_0}(|y-z|) dz \right) \\ &= \frac{3}{2} M_2 \left(\int_{|z|\leq|\tilde{z}_1|/2} G_{B_1}(\tilde{z}_1, z) J^{\varepsilon_0}(|y-z|) dz \right. \\ &\quad \left. + 2^d \int_{|\tilde{z}_1|/4<|w+\tilde{z}_1/2|<r/4} G_{B_1}(\tilde{z}_1, 2w+\tilde{z}_1) J^{\varepsilon_0}(|y-\tilde{z}_1-2w|) dw \right). \end{aligned} \quad (3.11)$$

Note that for any $y \in B_0^c$ and $|z| \leq |\tilde{z}_1|/2$, $|z-\tilde{z}_1| \geq |\tilde{z}_1| - |z| \geq |z|$ and $|y-\tilde{z}_1-\kappa z|/|y-z| \leq (|y|+|\tilde{z}_1|+\kappa|z|)/(|y|-|z|) \leq 4$. Thus

$$\begin{aligned} G_{B_1}(\tilde{z}_1, z) &\asymp |z-\tilde{z}_1|^{\alpha-d} \left(1 \wedge \frac{\delta_{B_1}(\tilde{z}_1)^{\alpha/2} \delta_{B_1}(z)^{\alpha/2}}{|z-\tilde{z}_1|^\alpha} \right) \\ &\leq |z|^{\alpha-d} \left(1 \wedge \frac{\delta_{B_1}(0)^{\alpha/2} \delta_{B_1}(z)^{\alpha/2}}{|z|^\alpha} \right) \asymp G_{B_1}(0, z), \end{aligned}$$

and

$$J^{\varepsilon_0}(|y-z|) \leq J^{\varepsilon_0}\left(\frac{1}{4}|y-\tilde{z}_1-\kappa z|\right) \leq 4^{d+\alpha} J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa z|).$$

It follows then that for any $y \in B_0^c$,

$$\int_{|z|\leq|\tilde{z}_1|/2} G_{B_1}(\tilde{z}_1, z) J^{\varepsilon_0}(|y-z|) dz \leq c_2 \int_{|z|\leq|\tilde{z}_1|/2} G_{B_1}(0, z) J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa z|) dz. \quad (3.12)$$

Note that for $y \in B_0^c$ and $|\tilde{z}_1|/4 < |w + \tilde{z}_1/2| < r/4$, $\delta_{B_1}(2w + \tilde{z}_1) = r/2 - |2w + \tilde{z}_1| \leq 2(r/2 - |w|) = 2\delta_{B_1}(w)$, and $|y - \tilde{z}_1 - \kappa w|/|y - \tilde{z}_1 - 2w| \leq (|y| + \kappa|w + \tilde{z}_1/2| + (1 - \kappa/2)|\tilde{z}_1|) / (|y| - |\tilde{z}_1 + 2w|) \leq 2$. Thus

$$\begin{aligned} G_{B_1}(\tilde{z}_1, 2w + \tilde{z}_1) &\asymp |2w|^\alpha \left(1 \wedge \frac{\delta_{B_1}(\tilde{z}_1)^{\alpha/2} \delta_{B_1}(2w + \tilde{z}_1)^{\alpha/2}}{|2w|^\alpha} \right) \\ &\lesssim |w|^{\alpha-d} \left(1 \wedge \frac{\delta_{B_1}(0)^{\alpha/2} \delta_{B_1}(w)^{\alpha/2}}{|w|^\alpha} \right) \asymp G_{B_1}(0, w), \end{aligned}$$

and

$$J^{\varepsilon_0}(|y - \tilde{z}_1 - 2w|) \leq J^{\varepsilon_0}(|y - \tilde{z}_1 - \kappa w|/2) \leq 2^{d+\alpha} J^{\varepsilon_0}(|y - \tilde{z}_1 - \kappa w|).$$

Thus for any $y \in B_0^c$,

$$\begin{aligned} &\int_{|\tilde{z}_1|/4 < |w + \tilde{z}_1/2| < r/4} G_{B_1}(\tilde{z}_1, 2w + \tilde{z}_1) J^{\varepsilon_0}(|y - \tilde{z}_1 - 2w|) dw \\ &\leq c_3 \int_{|\tilde{z}_1|/4 < |w + \tilde{z}_1/2| < r/4} G_{B_1}(0, w) J^{\varepsilon_0}(|y - \tilde{z}_1 - \kappa w|) dw \\ &\leq c_3 \int_{B_1} G_{B_1}(0, w) J^{\varepsilon_0}(|y - \tilde{z}_1 - \kappa w|) dw. \end{aligned} \tag{3.13}$$

Using (3.12) and (3.13), we can continue the estimates in (3.11) to get that for any $y \in B_0^c$

$$K_{B_1}^b(\tilde{z}_1, y) \leq c_4 \int_{B_1} G_{B_1}(0, z) J^{\varepsilon_0}(|y - \tilde{z}_1 - \kappa z|) dz \tag{3.14}$$

Combining (3.10) and (3.14), we get

$$K_{B_1}^b(\tilde{z}_1, y) \leq c_5 \kappa^{-\alpha} K_{\tilde{B}_1}^b(\tilde{z}_1, y), \quad \text{for } y \in B_0^c.$$

It follows that

$$\begin{aligned} I_1(\tilde{z}_1) &= \int_{B_0^c} u(y) K_{B_1}^b(\tilde{z}_1, y) dy \leq c_5 \kappa^{-\alpha} \int_{B_0^c} u(y) K_{\tilde{B}_1}^b(\tilde{z}_1, y) dy \\ &\leq c_5 \kappa^{-\alpha} \int_{\tilde{B}_1^c} u(y) K_{\tilde{B}_1}^b(\tilde{z}_1, y) dy = c_5 \kappa^{-\alpha} u(\tilde{z}_1). \end{aligned} \tag{3.15}$$

Consequently by (3.9) and (3.15) we have for all $k \geq 1$,

$$u_k(\tilde{z}_k) \leq I_k(\tilde{z}_k) \leq c_1 c_5 \kappa^{-\alpha} 2^{-k\alpha} u(\tilde{z}_1). \tag{3.16}$$

By the monotonicity of u_k in k , Theorem 1.3, (3.16) and Lemma 3.5, we conclude that for any $x \in D \cap B_k$ and $k \geq 1$

$$\frac{u_k(x)}{u(x)} \leq \frac{u_{k-1}(x)}{u(x)} \leq c_6 \frac{u_{k-1}(\tilde{z}_{k-1})}{u(\tilde{z}_{k-1})} \leq c_6 c_1 c_5 \kappa^{-\alpha} 2^{-(k-1)\alpha} \frac{u(\tilde{z}_1)}{u(\tilde{z}_{k-1})} \leq c_7 2^{-k\alpha}.$$

The proof is now complete. \square

For a Lipschitz open set D with characteristics (λ_0, R_0) , let $\kappa = \kappa(\lambda_0, R_0) \in (0, 1)$ so that D is κ -fat. For $z_0 \in \partial D$ and $r \in (0, 1]$, we use $A_r(z_0)$ to denote a point in D such that $B(A_r(z_0), \kappa r) \subset D \cap B(z_0, r)$.

The following lemma follows from Theorem 1.3 and Lemma 3.6 (in place of [2, Lemma 13 and Lemma 14]) in the same way as for the case of symmetric α -stable process in [2, Lemma 16]. We omit its proof here. See also [18, Lemma 3.4] for detailed computation of a similar result for symmetric Lévy processes.

Lemma 3.7. *Suppose Assumption 1 holds and D is a Lipschitz open set with characteristics (λ_0, R_0) . Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. There exist positive constants $\gamma_1 = \gamma_1(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2)$ and $C_{10} = C_{10}(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2)$ such that for every $z_0 \in \partial D$, $r \in (0, r_1/2)$ and all non-negative functions u, v that are regular harmonic in $D \cap B(z_0, 2r)$ and vanish in $D^c \cap B(z_0, 2r)$ with $u(A_r(z_0)) = v(A_r(z_0)) > 0$, we have*

- (i) $h(z_0) := \lim_{D \ni x \rightarrow z_0} u(x)/v(x)$ exists;
- (ii) $|\frac{u(x)}{v(x)} - h(z_0)| \leq C_{10} \left(\frac{|x - z_0|}{r} \right)^{\gamma_1}$ for $x \in D \cap B(z_0, r)$.

4 Gradient upper bound estimates

We now study gradient estimates for non-negative harmonic functions of X^b in open sets.

Lemma 4.1. *Suppose b is a bounded function satisfying (1.5), and that for every $x \in \mathbb{R}^d$, $J^b(x, y) \geq 0$ a.e. $y \in \mathbb{R}^d$. Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2, and $B = B(x_0, r)$ with $r \in (0, r_1]$. Then for every $x \in B$, $z \in \bar{B}^c$ and $1 \leq i \leq d$,*

$$\partial_{x_i} \int_B G_B(x, y) J^b(y, z) dy = \int_B \partial_{x_i} G_B(x, y) J^b(y, z) dy, \quad (4.1)$$

$$\partial_{x_i} \int_B \left(\int_B G_B(x, y) \mathcal{S}_y^b G_B^b(y, w) dy \right) J^b(w, z) dw = \int_B \left(\int_B \partial_{x_i} G_B(x, y) \mathcal{S}_y^b G_B^b(y, w) dy \right) J^b(w, z) dw, \quad (4.2)$$

and

$$\partial_{x_i} K_B^b(x, z) = \int_B \partial_{x_i} G_B(x, y) J^b(y, z) dy + \int_B \left(\int_B \partial_{x_i} G_B(x, y) \mathcal{S}_y^b G_B^b(y, w) dy \right) J^b(w, z) dw. \quad (4.3)$$

Proof. Without loss of generality we assume $i = d$. Fix $x \in B$ and $z \in \bar{B}^c$. We have

$$\sup_{y \in B} |J^b(y, z)| \leq \frac{\mathcal{A}(d, -\alpha)}{\delta_B(z)^{d+\alpha}} + \frac{\|b\|_\infty \mathcal{A}(d, -\beta)}{\delta_B(z)^{d+\beta}} < +\infty.$$

Thus (4.1) follows directly from [4, Lemma 5.2].

Let $g_z(y) := \int_B \mathcal{S}_y^b G_B^b(y, w) J^b(w, z) dw$ for $y \in B$. We have

$$\partial_{x_d} \int_B G_B(x, y) g_z(y) dy = \lim_{\lambda \rightarrow 0} \int_B \left[\frac{G_B(x + \lambda e_d, y) - G_B(x, y)}{\lambda} \right] g_z(y) dy. \quad (4.4)$$

To prove (4.2), we only need to show that the integrand in the right hand side of (4.4) is uniformly integrable on B in $\lambda \in (0, \delta_B(x)/2)$. Note that we have

$$G_B(x, y) = G(x, y) - \mathbb{E}_y[G(x, X_{\tau_B})] =: G(x, y) - H(x, y).$$

Thus

$$\begin{aligned} \frac{|G_B(x + \lambda e_d, y) - G_B(x, y)|}{\lambda} &\leq \frac{|G(x + \lambda e_d, y) - G(x, y)|}{\lambda} + \frac{|H(x + \lambda e_d, y) - H(x, y)|}{\lambda} \\ &=: I + II. \end{aligned}$$

Obviously by (2.7) we have

$$I \leq c_1 \left(|x + \lambda e_d - y|^{\alpha-1-d} + |x - y|^{\alpha-d-1} \right) \quad \text{for } y \in B,$$

for some positive constant $c_1 = c_1(d, \alpha)$. Since $H(x, y) = \mathbb{E}_y[G(x, X_{\tau_B})]$, by the mean-value theorem, there is a point x_λ in the line segment connecting x with $x + \lambda e_d$ so that

$$II = \partial_{x_d} H(x_\lambda, y) = \mathbb{E}_y[\partial_{x_d} G(x_\lambda, X_{\tau_B})] \leq c_2 \delta_B(x)^{\alpha-1-d}.$$

Thus for some positive constant $c_3 = c_3(d, \alpha, x)$, we have

$$\frac{|G_B(x + \lambda e_d, y) - G_B(x, y)|}{\lambda} \leq c_1 (|x + \lambda e_d - y| \wedge |x - y|)^{\alpha-d-1} + c_3. \quad (4.5)$$

Let $h(y) := \int_B h_B(y, w) dw$ for $y \in B$. Note that by Lemma 3.2 and the boundedness of $w \mapsto J^b(w, z)$ on B ,

$$|g_z(y)| \leq c_4 \int_B h_B(y, w) J^b(w, z) dw \leq c_5 h(y). \quad (4.6)$$

Thus by (4.5) and (4.6) the integrand in the right hand side of (4.4) is uniformly integrable on B in $\lambda \in (0, \delta_B(x)/2)$ if the following three conditions are true:

- (i) $\int_B h(y) dy < +\infty$;
- (ii) $\sup_{w \in B(x, \delta_B(x)/2)} \int_B h(y) |y - w|^{\alpha-1-d} dy < +\infty$;
- (iii) $\lim_{\varepsilon \downarrow 0} \sup_{w \in B(x, \delta_B(x)/2)} \int_{\{y \in B: |y-w| < \varepsilon\}} h(y) |y - w|^{\alpha-1-d} = 0$.

If $\alpha > 2\beta$, then for any $y \in B$,

$$h(y) \leq \int_{w \in B} |y - w|^{\alpha-\beta-d} dw \leq \int_{|u| < 2r} |u|^{\alpha-\beta-d} du < +\infty;$$

that is, $h(y)$ is bounded from above on B . Obviously (i)-(iii) hold for h . If $\alpha = 2\beta$, then

$$\begin{aligned} h(y) &= \int_{w \in B} |w - y|^{\beta-d} \left(1 \wedge \frac{\delta_B(w)^\beta}{|y - w|^\beta} \right) \left(1 \vee \log \frac{|w - y|}{\delta_B(y)} \right) dw \\ &= \int_{w \in B, |w-y| > c\delta_B(y)} |w - y|^{\beta/2-d} \frac{\delta_B(w)^\beta}{\delta_B(y)^{\beta/2}} \left(\frac{\delta_B(y)^{\beta/2}}{|w - y|^{\beta/2}} \log \frac{|w - y|}{\delta_B(y)} \right) dw \end{aligned}$$

$$\begin{aligned}
& + \int_{w \in B, |w-y| \leq \varepsilon \delta_B(y)} |w-y|^{\beta-d} dw \\
& \lesssim \delta_B(y)^{-\beta/2} + 1.
\end{aligned} \tag{4.7}$$

Using this upper bound, it is easy to check h satisfies (i) and (ii). As for (iii), note that $\delta_B(w) \geq \delta_B(x)/2$ for every $w \in B(x, \delta_B(x)/2)$. Consider an arbitrary $\varepsilon \in (0, \delta_B(x)/4)$. Then $B(w, \varepsilon) \subset B$ and $\delta_B(y) \geq \delta_B(x)/4$ for every $y \in B(w, \varepsilon)$. We have by (4.7),

$$\begin{aligned}
\int_{|y-w| < \varepsilon} h(y) |y-w|^{\alpha-1-d} dy & \lesssim \int_{|y-w| < \varepsilon} (\delta_B(y)^{-\beta/2} + 1) |y-w|^{\alpha-1-d} dy \\
& \lesssim \int_{|y-w| < \varepsilon} (\delta_B(x)^{-\beta/2} + 1) |y-w|^{\alpha-1-d} dy.
\end{aligned}$$

Thus condition (iii) is implied by the fact that

$$\lim_{\varepsilon \downarrow 0} \sup_{w \in B(x, \delta_B(x)/2)} \int_{|y-w| < \varepsilon} (\delta_B(x)^{-\beta/2} + 1) |y-w|^{\alpha-1-d} dy = 0.$$

When $\alpha < 2\beta$, similar to (4.7) we have

$$h(y) \asymp \int_{w \in B} |w-y|^{\beta-d} \left(1 \wedge \frac{\delta_B(w)^\beta}{|y-w|^\beta} \right) \left(1 \vee \frac{|y-w|^{\beta-\alpha/2}}{\delta_B(y)^{\beta-\alpha/2}} \right) dw \lesssim \delta_B(y)^{\beta-\alpha/2} + 1.$$

By a similar calculations as in the case $\alpha = 2\beta$, we can show that (i)-(iii) hold for h . This completes the proof. \square

Lemma 4.2. *Under Assumption 1, there exists a constant $C_{11} = C_{11}(d, \alpha, \beta, M_1, M_2) > 0$ such that for every $B = B(x_0, 1)$ and $1 \leq i \leq d$,*

$$\int_B |\partial_{x_i} G_B(x, y)| J^b(y, z) dy \leq C_{11} \int_B G_B(x_0, y) J^b(y, z) dy \quad \text{for } x \in B(x_0, 1/4) \text{ and } z \in \bar{B}^c. \tag{4.8}$$

Proof. Without loss of generality, we assume $x_0 = 0$ and $i = d$. For every $|x| < 1/4$ and $|y| < 1$, we have $|x-y| \wedge \delta_B(x) \asymp |x-y|$. Thus by (3.1),

$$\begin{aligned}
\int_B |\partial_{x_d} G_B(x, y)| J^b(y, z) dy & \lesssim \int_{|y| < 1} \frac{G_B(x, y)}{|x-y| \wedge \delta_B(x)} J^b(y, z) dy \\
& \asymp \int_{|y| < 1} \frac{G_B(x, y)}{|x-y|} J^b(y, z) dy \\
& = \left(\int_{1/2 \leq |y| < 1} + \int_{|y| < 1/2} \right) \frac{G_B(x, y)}{|x-y|} J^b(y, z) dy \\
& =: I(x, z) + II(x, z).
\end{aligned}$$

For $1/2 \leq |y| < 1$ and $|x| < 1/4$, we have $|x-y| \asymp |y| \asymp 1$ and $\delta_B(x) \asymp 1$. Thus

$$G_B(x, y) \asymp |y|^{\alpha-d} \left(1 \wedge \frac{\delta_B(y)^{\alpha/2}}{|y|^\alpha} \right) \asymp G_B(0, y),$$

and consequently

$$I(x, z) \asymp \int_{1/2 \leq |y| < 1} G_B(0, y) J^b(y, z) dy. \quad (4.9)$$

For every $|y| < 1/2$, $|x| < 1/4$ and $|z| > 1$, we have $|z - y| > 1/2$, $|z - y| \asymp |z - y + x|$ and $\delta_B(y) = 1 - |y| \asymp 1 - |y - x| \asymp 1$. Thus by (1.6)

$$\begin{aligned} II(x, z) &\stackrel{c_1(M_2)}{\asymp} \int_{|y| < 1/2} |x - y|^{-d+\alpha-1} \left(1 \wedge \frac{\delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2}}{|x - y|^\alpha} \right) J^{\varepsilon_0}(|z - y|) dy \\ &\asymp \int_{|y| < 1/2} |x - y|^{-d+\alpha-1} \left(1 \wedge \frac{\delta_B(x)^{\alpha/2} (1 - |y - x|)^{\alpha/2}}{|x - y|^\alpha} \right) J^{\varepsilon_0}(|z - y + x|) dy \\ &\asymp \int_{|w+x| < 1/2} |w|^{-d+\alpha-1} \left(1 \wedge \frac{(1 - |w|)^{\alpha/2}}{|w|^\alpha} \right) J^{\varepsilon_0}(|z - w|) dw \\ &\leq \int_{|w| < 3/4} |w|^{-d+\alpha-1} \left(1 \wedge \frac{(1 - |w|)^{\alpha/2}}{|w|^\alpha} \right) J^{\varepsilon_0}(|z - w|) dw \\ &\stackrel{c_2(M_2)}{\asymp} \int_{|w| < 3/4} |w|^{-d+\alpha-1} \left(1 \wedge \frac{(1 - |w|)^{\alpha/2}}{|w|^\alpha} \right) J^b(w, z) dw \\ &=: g_1(z). \end{aligned}$$

For every $z > 1$, let

$$g_2(z) := \int_{|w| < 3/4} |w|^{\alpha-d} \left(1 \wedge \frac{(1 - |w|)^{\alpha/2}}{|w|^\alpha} \right) J^b(w, z) dw.$$

Obviously

$$g_2(z) \asymp \int_{|w| < 3/4} G_B(0, w) J^b(w, z) dw \leq \int_B G_B(0, w) J^b(w, z) dw. \quad (4.10)$$

Note that $J^{\varepsilon_0}(|x - y|)$ is non-increasing in $|x - y|$. Thus by (1.6)

$$\sup_{|z| > 1} \frac{g_1(z)}{g_2(z)} \leq \sup_{|z| > 1} \frac{M_2 J^{\varepsilon_0}(|z| - 3/4) \int_{|w| < 3/4} |w|^{\alpha-d-1} \left(1 \wedge \frac{(1 - |w|)^{\alpha/2}}{|w|^\alpha} \right) dw}{M_2^{-1} J^{\varepsilon_0}(|z| + 3/4) \int_{|w| < 3/4} |w|^{\alpha-d} \left(1 \wedge \frac{(1 - |w|)^{\alpha/2}}{|w|^\alpha} \right) dw} \leq M < +\infty, \quad (4.11)$$

where $M = M(d, \alpha, \beta, M_2) > 0$. Thus by (4.10) and (4.11) we prove that

$$II(x, z) \stackrel{c_3(M_2)}{\lesssim} \int_B G_B(0, w) J^b(z, w) dw \quad \text{for } |x| < 1/4, |z| > 1. \quad (4.12)$$

Therefore (4.8) follows from (4.9) and (4.12). \square

Recall the definition of $r_D(x, y)$ and $h_D(x, y)$ from (1.2) and (3.2), respectively.

Lemma 4.3. *For $B = B(0, 1)$, there exists a constant $C_{12} = C_{12}(d, \alpha, \beta) > 0$ such that for every $1 \leq i \leq d$, $|x| < 1/4$ and $|w| < 1$,*

$$\int_B |\partial_{x_i} G_B(x, y)| h_B(y, w) dy \leq C_{12} |x - w|^{-d+(\alpha-1) \wedge (\alpha-\beta)} \delta_B(w)^{\alpha/2}.$$

Proof. Without loss of generality, we assume $i = d$. For any $|x| < 1/4$ and $|y| < 1$, we have $\delta_B(x) \asymp 1$ and $|x - y| \wedge \delta_B(x) \asymp |x - y|$. Thus by (3.1),

$$\begin{aligned} \int_B |\partial_{x_d} G_B(x, y)| h_B(y, w) dy &\lesssim \int_{|y| < 1} \frac{G_B(x, y)}{|x - y| \wedge \delta_B(x)} h_B(y, w) dy \\ &\asymp \int_{|y| < 1} \frac{G_B(x, y)}{|x - y|} h_B(y, w) dy. \end{aligned} \quad (4.13)$$

We note that for $|x| < 1/4$ and $|y| < 1$,

$$G_B(x, y) \asymp |x - y|^{\alpha-d} \left(1 \wedge \frac{\delta_B(x)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \left(1 \wedge \frac{\delta_B(y)^{\alpha/2}}{|x - y|^{\alpha/2}} \right) \asymp |x - y|^{\alpha-d} \frac{\delta_B(y)^{\alpha/2}}{r_B(x, y)^\alpha}, \quad (4.14)$$

and $r_B(x, y) \geq \delta_B(x) \geq 3/4$. Now we calculate the integral in (4.13) using (4.14) and the explicit formula of $h_B(y, w)$. If $\alpha > 2\beta$, we have

$$\begin{aligned} (4.13) &\asymp \int_{|y| < 1} \frac{\delta_B(w)^{\alpha/2}}{|x - y|^{d-\alpha+1} |w - y|^{d-\alpha+\beta}} \frac{\delta_B(y)^{\alpha/2}}{r_B(x, y)^\alpha r_B(y, w)^{\alpha/2}} dy \\ &\lesssim \delta_B(w)^{\alpha/2} \int_{|y| < 1} |x - y|^{-d+\alpha-1} |y - w|^{-d+\alpha-\beta} dy \\ &\lesssim |x - w|^{-d+(\alpha-1) \wedge (\alpha-\beta)} \delta_B(w)^{\alpha/2} \\ &\leq |x - w|^{-d+(\alpha-1) \wedge (\alpha-\beta)} \delta_B(w)^\beta. \end{aligned} \quad (4.15)$$

If $\alpha = 2\beta$, we have by (2.10) and (3.2)

$$\begin{aligned} (4.13) &\asymp \int_{|y| < 1} |x - y|^{2\beta-1-d} \frac{\delta_B(y)^\beta}{r_B(x, y)^{2\beta}} |y - w|^{\beta-d} \frac{\delta_B(w)^\beta}{r_B(y, w)^\beta} \left(1 \vee \log \frac{|y - w|}{\delta_B(y)} \right) dy \\ &= \int_{|y| < 1, |y-w| \leq e\delta_B(y)} \frac{\delta_B(w)^\beta}{|x - y|^{d+1-2\beta} |y - w|^{d-\beta}} \frac{\delta_B(y)^\beta}{r_B(x, y)^{2\beta} r_B(y, w)^\beta} dy \\ &\quad + \int_{|y| < 1, |y-w| > e\delta_B(y)} \frac{\delta_B(w)^\beta}{|x - y|^{d+1-2\beta} |w - y|^{d-\beta}} \frac{|y - w|^{\beta/2} \delta_B(y)^{\beta/2}}{r_B(x, y)^{2\beta} r_B(y, w)^\beta} \left(\frac{\delta_B(y)^{\beta/2}}{|y - w|^{\beta/2}} \log \frac{|y - w|}{\delta_B(y)} \right) dy \\ &\lesssim \delta_B(w)^\beta \int_{|y| < 1, |y-w| \leq e\delta_B(y)} |x - y|^{2\beta-1-d} |y - w|^{\beta-d} dy \\ &\quad + \delta_B(w)^\beta \int_{|y| < 1, |y-w| > e\delta_B(y)} |x - y|^{2\beta-1-d} |y - w|^{2\beta-d} dy \\ &\lesssim \delta_B(w)^\beta |x - w|^{-d+2\beta-1} = |x - w|^{-d+(\alpha-1) \wedge (\alpha-\beta)} \delta_B(w)^\beta. \end{aligned} \quad (4.16)$$

If $\alpha < 2\beta$, we have

$$\begin{aligned} (4.13) &\asymp \int_{|y| < 1} |x - y|^{-d+\alpha-1} \frac{\delta_B(y)^{\alpha/2}}{r_B(x, y)^\alpha} |y - w|^{-d+\alpha-\beta} \frac{\delta_B(w)^{\alpha/2}}{r_B(y, w)^{\alpha/2}} \left(1 \vee \frac{|y - w|^{\beta-\alpha/2}}{\delta_B(y)^{\beta-\alpha/2}} \right) dy \\ &\leq \int_{|y| < 1, |y-w| \geq \delta_B(y)} \frac{\delta_B(w)^{\alpha/2}}{|x - y|^{d-\alpha+1} |w - y|^{d-\alpha+\beta}} \frac{|w - y|^{\beta-\alpha/2} \delta_B(y)^{\alpha-\beta}}{r_B(y, w)^{\alpha/2}} dy \end{aligned}$$

$$\begin{aligned}
& + \int_{|y|<1, |y-w|<\delta_B(y)} \frac{\delta_B(w)^{\alpha/2}}{|x-y|^{d-\alpha+1}|w-y|^{d-\alpha+\beta}} \frac{\delta_B(y)^{\alpha/2}}{r_B(x,y)^\alpha r_B(y,w)^{\alpha/2}} dy \\
& \lesssim \delta_B(w)^{\alpha/2} \int_{|y|<1, |y-w|\geq\delta_B(y)} |x-y|^{-d+\alpha-1} |y-w|^{-d+\alpha-\beta} dy \\
& \quad + \delta_B(w)^{\alpha/2} \int_{|y|<1, |y-w|<\delta_B(y)} |x-y|^{-d+\alpha-1} |y-w|^{-d+\alpha-\beta} dy \\
& = \delta_B(w)^{\alpha/2} \int_{|y|<1} |x-y|^{-d+\alpha-1} |y-w|^{-d+\alpha-\beta} dy \\
& \lesssim \delta_B(w)^{\alpha/2} |x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)} \int_{|y|<1} (|x-y|^{\alpha-1-(\alpha-1)\wedge(\alpha-\beta)} + |y-w|^{\alpha-\beta-(\alpha-1)\wedge(\alpha-\beta)}) dy \\
& \lesssim |x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)} \delta_B(w)^{\alpha/2}. \tag{4.17}
\end{aligned}$$

Lemma 4.3 follows from (4.16), (4.17) and (4.15). \square

Lemma 4.4. *Under Assumption 1, there exists a constant $C_{13} = C_{13}(d, \alpha, \beta, M_1, M_2) > 0$ such that for $B = B(x_0, 1)$, $1 \leq i \leq d$, $x \in B(x_0, 1/4)$ and $z \in \bar{B}^c$,*

$$\int_B \left[\int_B |\partial_{x_i} G_B(x, y) \mathcal{S}_y^b G_B^b(y, w)| dy \right] J^b(w, z) dw \leq C_{13} \int_B G_B(x_0, w) J^b(w, z) dw. \tag{4.18}$$

Proof. Without loss of generality, we assume that $x_0 = 0$ and $i = d$. Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. By Lemma 3.2 and the scaling property, we have for $y, w \in B$,

$$|\mathcal{S}_y^b G_B^b(y, w)| = r_1^d |\mathcal{S}_{y_1}^{b_{r_1}} G_{r_1 B}^{b_{r_1}}(r_1 y, r_1 w)| \leq c_1 r_1^d |h_{r_1 B}(r_1 y, r_1 w)| = c_1 r_1^{\alpha-\beta} h_B(y, w) \leq c_1 h_B(y, w).$$

Here $c_1 = c_1(d, \alpha, \beta, M_1) > 0$. Hence to prove (4.18), it suffices to prove that for $x \in B(0, 1/4)$ and $z \in \bar{B}^c$,

$$\int_B \left[\int_B |\partial_{x_d} G_B(x, y)| h_B(y, w) dy \right] J^b(w, z) dw \leq c_2 \int_B G_B(x_0, w) J^b(w, z) dw \tag{4.19}$$

for some $c_2 = c_2(d, \alpha, \beta, M_1, M_2) > 0$. By Lemma 4.3, we have

$$\begin{aligned}
& \int_B \left[\int_B |\partial_{x_d} G_B(x, y)| h_B(y, w) dy \right] J^b(w, z) dw \\
& \lesssim \int_{|w|<1} \delta_B(w)^{\alpha/2} |w-x|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^b(w, z) dw \\
& = \int_{|w|\leq 1/2} + \int_{1/2<|w|<1} \delta_B(w)^{\alpha/2} |w-x|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^b(w, z) dw \\
& =: I(x, z) + II(x, z).
\end{aligned}$$

Fix $|x| < 1/4$ and $|z| > 1$. For $1/2 < |w| < 1$, we have $|w-x| \asymp |w| \asymp 1$, and consequently $G_B(0, w) \asymp \delta_B(w)^{\alpha/2}$. Thus

$$II(x, z) \asymp \int_{1/2<|w|<1} \delta_B(w)^{\alpha/2} J^b(w, z) dw \asymp \int_{1/2<|w|<1} G_B(0, w) J^b(w, z) dw. \tag{4.20}$$

For any $|w| \leq 1/2$, we have $1/2 \leq \delta_B(w) \leq 1$, $|z - w| \geq 1/2$ and $|z - w + x| \asymp |z - w|$. Hence by (1.6)

$$\begin{aligned}
I(x, z) &\stackrel{M_2}{\asymp} \int_{|w| \leq 1/2} |w - x|^{-d+(\alpha-1) \wedge (\alpha-\beta)} J^{\varepsilon_0}(|z - w|) dw \\
&\asymp \int_{|w| \leq 1/2} |w - x|^{-d+(\alpha-1) \wedge (\alpha-\beta)} J^{\varepsilon_0}(|z - w + x|) dw \\
&= \int_{|v+x| \leq 1/2} |v|^{-d+(\alpha-1) \wedge (\alpha-\beta)} J^{\varepsilon_0}(|z - v|) dv \\
&\leq \int_{|v| \leq 3/4} |v|^{-d+(\alpha-1) \wedge (\alpha-\beta)} J^{\varepsilon_0}(|z - v|) dv =: g_1(z). \tag{4.21}
\end{aligned}$$

We first consider the case $|z| > 2$. Let $g_2(z) := \int_{|v| \leq 3/4} |v|^{\alpha-d} J^{\varepsilon_0}(|z - v|) dv$. Note that for any $|v| \leq 3/4$, we have $G_B(0, v) \asymp |v|^{\alpha-d}$. Thus

$$g_2(z) \asymp \int_{|v| \leq 3/4} G_B(0, v) J^{\varepsilon_0}(|z - v|) dv \leq M_2 \int_{|v| \leq 3/4} G_B(0, v) J^b(v, z) dv. \tag{4.22}$$

In addition since $J^{\varepsilon_0}(|y|)$ is non-increasing in $|y|$, we have

$$\sup_{|z| > 2} \frac{g_1(z)}{g_2(z)} \leq \sup_{|z| > 2} \frac{J^{\varepsilon_0}(|z| - 3/4) \int_{|v| \leq 3/4} |v|^{-d+(\alpha-1) \wedge (\alpha-\beta)} dv}{J^{\varepsilon_0}(|z| + 3/4) \int_{|v| \leq 3/4} |v|^{\alpha-d} dv} \leq M < +\infty, \tag{4.23}$$

where $M = M(d, \alpha, \beta) > 0$. Therefore by (4.21), (4.22) and (4.23) we have

$$I(x, z) \stackrel{c_3(M_2)}{\lesssim} \int_B G_B(0, w) J^b(w, z) dw \quad \text{for } |x| < 1/4 \text{ and } |z| > 2. \tag{4.24}$$

On the other hand if $1 < |z| \leq 2$, we have $0 < \delta_B(z) \leq 1$, and by (2.11)

$$\int_B G_B(0, w) J^b(w, z) dw \geq M_2^{-1} \int_B G_B(0, w) J(w, z) dw = M_2^{-1} K_B(0, z) \asymp M_2^{-1} \delta_B(z)^{-\alpha/2} \geq M_2^{-1}. \tag{4.25}$$

Note that $|z - w| \geq 1/4$ for any $|w| \leq 3/4$. Thus

$$g_1(z) \leq J^{\varepsilon_0}(1/4) \int_{|w| \leq 3/4} |w|^{-d+(\alpha-1) \wedge (\alpha-\beta)} dw \lesssim 1. \tag{4.26}$$

Thus by (4.21), (4.25) and (4.26) we have

$$I(x, z) \stackrel{c_4(M_2)}{\lesssim} \int_B G_B(0, w) J^b(w, z) dw \quad \text{for } |x| < 1/4 \text{ and } 1 < |z| \leq 2. \tag{4.27}$$

Now (4.19) follows from (4.27) and (4.20). \square

Theorem 4.5. *Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. Under Assumption 1, there exists a constant $C_{14} = C_{14}(d, \alpha, \beta, M_1, M_2) > 0$ such that for every ball $B_r = B(x_0, r)$ with radius $r \in (0, r_1]$ and $1 \leq i \leq d$,*

$$|\partial_{x_i} K_{B_r}^b(x, z)| \leq \frac{C_{14}}{r} K_{B_r}^b(x_0, z), \quad \text{for } x \in B(x_0, r/4) \text{ and } z \in \bar{B}_r^c. \tag{4.28}$$

Proof. Let $\lambda := 1/r \geq 1/r_1 \geq 1$ and define $b_\lambda(x, z) = \lambda^{\beta-\alpha} b(\lambda^{-1}x, \lambda^{-1}z)$. Observe that $\|b_\lambda\|_\infty = r^{\alpha-\beta} \|b\|_\infty \leq r_1^{\alpha-\beta} M_1 \leq M_1$. By the scaling properties (2.4) and (2.6), $b_\lambda(x, z)$ satisfies Assumption 1 and it suffices to show that for the ball $B = B(x_0, 1)$,

$$|\partial_{x_i} K_B^{b_\lambda}(x, z)| \leq C_{14} K_B^{b_\lambda}(x_0, z) \quad \text{for } x \in B(x_0, 1/4) \text{ and } z \in \bar{B}^c. \quad (4.29)$$

We know from [13, Lemma 4.10] that

$$G_B^{b_\lambda}(x, y) = G_B(x, y) + \int_B G_B(x, z) \mathcal{S}_z^{b_\lambda} G_B^{b_\lambda}(z, y) dz \quad \text{for } x, y \in B.$$

Thus by (2.3), for $i = 1, \dots, d$, every $x \in B$, and $z \in \bar{B}^c$,

$$\partial_{x_i} K_B^{b_\lambda}(x, z) = \partial_{x_i} \int_B G_B(x, y) J^{b_\lambda}(y, z) dy + \partial_{x_i} \int_B \left(\int_B G_B(x, y) \mathcal{S}_z^{b_\lambda} G_B^{b_\lambda}(y, w) dy \right) J^{b_\lambda}(w, z) dw.$$

Thus by Lemma 4.1

$$\begin{aligned} |\partial_{x_i} K_B^{b_\lambda}(x, z)| &\leq \int_B |\partial_{x_i} G_B(x, y)| J^{b_\lambda}(y, z) dy \\ &\quad + \int_B \left(\int_B |\partial_{x_i} G_B(x, y) \mathcal{S}_z^{b_\lambda} G_B^{b_\lambda}(y, w)| dy \right) J^{b_\lambda}(w, z) dw. \end{aligned}$$

On the other hand, by (3.3) and (2.5), we have

$$\frac{1}{2} G_B(x, y) \leq G_B^{b_\lambda}(x, y) \leq \frac{3}{2} G_B(x, y) \quad \text{for } x, y \in B.$$

Thus

$$K_B^{b_\lambda}(x, z) = \int_B G_B^{b_\lambda}(x, y) J^{b_\lambda}(y, z) dy \geq \frac{1}{2} \int_B G_B(x, y) J^{b_\lambda}(y, z) dy \quad \text{for } z \in \bar{B}^c.$$

Now (4.29) is implied by (3.3) and Lemmas 4.2-4.4. This completes the proof of the theorem. \square

Lemma 4.6. *Suppose Assumption 1 holds and f is regular harmonic with respect to X^b in $B(x, r)$ for some $x \in \mathbb{R}^d$ and $r \in (0, r_1]$. Then $\partial_{x_i} f(x)$ exists for every $1 \leq i \leq d$ and*

$$\partial_{x_i} f(x) = \int_{B(x, r)^c} f(z) \partial_{x_i} K_{B(x, r)}^b(x, z) dz. \quad (4.30)$$

Proof. Recall that e_i is the unit vector along the positive x_i -axis. Choose $\varepsilon > 0$ sufficiently small so that $x + \varepsilon e_i \in B(x, r/4)$. By the regular harmonicity of f , we have

$$\frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon} = \int_{B(x, r)^c} f(z) \left[\frac{K_{B(x, r)}^b(x + \varepsilon e_i, z) - K_{B(x, r)}^b(x, z)}{\varepsilon} \right] dz.$$

Therefore (4.30) follows from (4.28) and the dominated convergence theorem. \square

Proof of Theorem 1.4: Let $x \in D$ and $0 < r < (\delta_D(x) \wedge r_1)/2$. Note that under our assumption, f is regular harmonic in $B(x, r)$ with respect to X^b . By (4.30) and (4.28), we have

$$|\partial_{x_i} f(x)| \leq \int_{B(x, r)^c} f(z) |\partial_{x_i} K_{B(x, r)}^b(x, z)| dz$$

$$\begin{aligned}
&\leq \frac{C_{14}}{r} \int_{B(x,r)^c} f(z) K_{B(x,r)}^b(x, z) dz \\
&= \frac{C_{14}}{r} f(x) \rightarrow \frac{2C_{14}}{r_1 \wedge \delta_D(x)} f(x) \quad \text{as } r \uparrow (r_1 \wedge \delta_D(x))/2.
\end{aligned}$$

□

5 Gradient lower bound estimate

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we write $x = (\tilde{x}, x_d)$, where $\tilde{x} = (x_1, \dots, x_{d-1})$. In this section, we fix a Lipschitz function $\Gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ with Lipschitz constant λ_0 so that $|\Gamma(\tilde{x}) - \Gamma(\tilde{y})| \leq \lambda_0 |\tilde{x} - \tilde{y}|$ for all $\tilde{x}, \tilde{y} \in \mathbb{R}^{d-1}$. Put $\rho(x) := x_d - \Gamma(\tilde{x})$. Unless stated otherwise, D denotes the special Lipschitz open set defined by $D = \{x \in \mathbb{R}^d : \rho(x) > 0\}$. When $x \in D$, $\rho(x)$ serves as the vertical distance from $x \in D$ to ∂D , and it satisfies

$$\rho(x)/\sqrt{1 + \lambda_0^2} \leq \delta_D(x) \leq \rho(x) \quad \text{for } x \in D. \quad (5.1)$$

We define the “box” $D^+(x, h, r) := \{y \in \mathbb{R}^d : 0 < \rho(y) < h, |\tilde{x} - \tilde{y}| < r\}$, and the “inverted box” $D^-(x, h, r) := \{y \in \mathbb{R}^d : -h < \rho(y) \leq 0, |\tilde{x} - \tilde{y}| < r\}$, where $x \in \mathbb{R}^d$ and $h, r > 0$.

Lemma 5.1. *Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. Suppose Assumption 1 holds, $z_0 \in \partial D$ and $r \in (0, r_1/2]$. Let $A_{z_0} \in D$ be such that $\rho(A_{z_0}) = |A_{z_0} - z_0| = r/2$. Then there exist positive constants $C_{15} = C_{15}(d, \alpha, \beta, \lambda_0, M_1, M_2)$ and $\gamma_2 = \gamma_2(d, \alpha, \beta, \lambda_0, M_1, M_2)$ such that for every non-negative function u which is harmonic in $D \cap B(z_0, 2r)$ and vanishes in $D^c \cap B(z_0, 2r)$, we have*

$$\frac{u(x)}{u(A_{z_0})} \geq C_{15} \left(\frac{\rho(x)}{\rho(A_{z_0})} \right)^{\alpha - \gamma_2} \quad \text{for } x \in D \cap B(z_0, r).$$

Proof. Note that by Lemma 3.2 and Assumption 1, we have for every $x \in \mathbb{R}^d$ and $y \in \overline{B(x, r)}^c$,

$$K_{B(x, r)}^b(x, y) \stackrel{M_2}{\asymp} \int_{B(x, r)} G_{B(x, r)}(x, z) J^{\varepsilon_0}(z, y) dz \asymp K_{B(x, r)}^{\varepsilon_0}(x, y). \quad (5.2)$$

Lemma 5.1 follows from (5.2), the uniform Harnack inequality (Theorem 3.4) and a standard argument of induction in the same way as for the case of symmetric α -stable process in [2, Lemma 5] (see also [4, Lemma 4.2]). We omit the details here. □

In the remaining of this section, we assume that Assumption 1 and Assumption 2 with $i = d$ hold. In this case, the jumping kernel $J^b(x, y)$ of the process X^b satisfies that for every $x \in \mathbb{R}^d$,

$$J^b(x, y) = \frac{\mathcal{A}(d, -\alpha)}{|y - x|^{d+\alpha}} + \mathcal{A}(d, -\beta) \frac{\varphi(\tilde{x})\psi(|y - x|)}{|y - x|^{d+\beta}} =: j^b(\tilde{x}, |y - x|) \quad \text{a.e. } y \in \mathbb{R}^d. \quad (5.3)$$

We note that by condition (1.7) of Assumption 2, $j^b(\tilde{x}, |z|)$ is non-increasing in $|z|$ for every $\tilde{x} \in \mathbb{R}^{d-1}$. Fix $z_0 \in \partial D$, $r \in (0, r_1]$. We define $D^+ := D^+(z_0, 4r\sqrt{1 + \lambda_0^2}, 2r) \supset D \cap B(z_0, 2r)$, and

$$g_{b, r}(x) := \mathbb{P}_x \left(X_{\tau_{D^+}}^b \notin D^-(z_0, \infty, 2r) \right) \quad \text{for } x \in \mathbb{R}^d.$$

Clearly, $g_{b, r}$ is regular harmonic in D^+ with respect to X^b , $g_{b, r}(x) = 0$ in $D^-(z_0, \infty, 2r)$, and $g_{b, r}(x) = 1$ in $(D^+ \cup D^-(z_0, \infty, 2r))^c$.

Lemma 5.2. *The function $g_{b,r}(x)$ is non-decreasing in x_d .*

Proof. Note that $g_{b,r}(x) = 1 - \mathbb{P}_x \left(X_{\tau_{D^+}}^b \in D^-(z_0, \infty, 2r) \right)$ for every $x \in \mathbb{R}^d$. Take $x, y \in D^+$ such that $\tilde{x} = \tilde{y}$ and $y_d < x_d$. Consider the process (X_t^b, \mathbb{P}_x) starting from x (i.e. $\mathbb{P}_x(X_0^b = x) = 1$). For every $t \geq 0$, define

$$Y_t^b := X_t^b - (x_d - y_d)e_d.$$

Then (Y_t^b, \mathbb{P}_x) is a Markov process starting from y . Let $\mathcal{S}(\mathbb{R}^d)$ denote the totality of tempered functions on \mathbb{R}^d . For every $f \in \mathcal{S}(\mathbb{R}^d)$, if we define $f_d(z) := f(z - (x_d - y_d)e_d)$ for $z \in \mathbb{R}^d$, then

$$\mathcal{L}^b f(x - (x_d - y_d)e_d) = \mathcal{L}^b f_d(x) = \lim_{\varepsilon \rightarrow 0} \int_{|y-x|>\varepsilon} (f_d(y) - f_d(x)) j^b(\tilde{x}, |y-x|) dy \text{ for } x \in \mathbb{R}^d.$$

Thus

$$f(Y_t^b) - f(Y_0^b) - \int_0^t \mathcal{L}^b f(Y_s^b) ds = f_d(X_t^b) - f_d(X_0^b) - \int_0^t \mathcal{L}^b f_d(X_s^b) ds$$

is a \mathbb{P}_x -martingale. We know from [12, Theorem 1.3] that the solution of the martingale problem $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ with initial value y is unique. Hence (Y_t^b, \mathbb{P}_x) has the same distribution as (X_t^b, \mathbb{P}_y) . Consider the trajectory ω of X_t^b starting from x . If ω exits D^+ by going into $D^-(0, +\infty, 2r)$, then so does $\omega - (x_d - y_d)e_d$ which is the trajectory of Y_t^b starting from y . Hence

$$\mathbb{P}_x \left(X_{\tau_{D^+}}^b \in D^-(z_0, \infty, 2r) \right) \leq \mathbb{P}_x \left(Y_{\tau_{D^+}}^b \in D^-(z_0, \infty, 2r) \right) = \mathbb{P}_y \left(X_{\tau_{D^+}}^b \in D^-(z_0, \infty, 2r) \right).$$

This completes the proof. \square

Lemma 5.3. *Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. There are constants*

$C_{16} = C_{16}(d, \alpha, \beta, \lambda_0, M_1, M_2) > 0$ and $r_2 = r_2(d, \alpha, \beta, \lambda_0, M_1, M_2) \in (0, r_1]$ such that for every $z_0 \in \partial D$ and $r \in (0, r_2]$,

$$\partial_{x_d} g_{b,r}(x) \geq C_{16} \frac{g_{b,r}(x)}{\delta_D(x)} \text{ for } x \in D \cap B(z_0, r/2).$$

Proof. Without loss of generality we assume $z_0 = 0$. Let $r_2 \in (0, r_1]$ to be specified later. For $r \in (0, r_2]$, fix $x \in D \cap B(0, r/2)$. By (5.1), $r_0 := \rho(x)/2\sqrt{1+\lambda_0^2} \leq \delta_D(x)/2 \leq r/4 \leq r_2/4$. Set

$$\hat{x} = x + 2\rho(x)e_d \quad \text{and} \quad \check{x} = x - 2\rho(x)e_d.$$

Observe that $B(x, r_0), B(\hat{x}, r_0) \subset D^+$ and $B(\check{x}, r_0) \subset D^-(0, +\infty, 2r)$. By (4.30) and (4.3), we have

$$\begin{aligned} \partial_{x_d} g_{b,r}(x) &= \int_{\overline{B(x, r_0)}^c} g_{b,r}(z) \partial_{x_d} K_{B(x, r_0)}^b(x, z) dz \\ &\geq \int_{\overline{B(x, r_0)}^c} g_{b,r}(z) \left[\int_{B(x, r_0)} \partial_{x_d} G_{B(x, r_0)}(x, y) J^b(y, z) dy \right] dz \\ &\quad - \int_{\overline{B(x, r_0)}^c} g_{b,r}(z) \left(\int_{B(x, r_0) \times B(x, r_0)} \left| \partial_{x_d} G_{B(x, r_0)}(x, y) S_y^b G_{B(x, r_0)}^b(y, w) \right| J^b(w, z) dy dw \right) dz. \end{aligned} \tag{5.4}$$

Let $\lambda := 1/r_0$ and $B_1 := \lambda B(x, r_0)$. By scaling property, Lemma 4.4 and Lemma 3.2, we have

$$\begin{aligned}
& \int_{B(x, r_0)} \int_{B(x, r_0)} \left| \partial_{x_d} G_{B(x, r_0)}(x, y) S_y^b G_{B(x, r_0)}^b(y, w) \right| J^b(w, z) dy dw \\
&= \lambda^{d+1} \int_{B_1} \int_{B_1} \left| \partial_{x_d} G_{B_1}(\lambda x, y) S_y^{b\lambda} G_{B_1}^{b\lambda}(y, w) \right| J^{b\lambda}(w, \lambda z) dy dw \\
&\leq c_1 \lambda^{d+1-\alpha+\beta} \int_{B_1} G_{B_1}(\lambda x, w) J^{b\lambda}(w, \lambda z) dw \\
&= c_1 \lambda^{1-\alpha+\beta} \int_{B(x, r_0)} G_{B(x, r_0)}(x, v) J^b(v, z) dv \\
&\leq 2c_1 \lambda^{1-\alpha+\beta} \int_{B(x, r_0)} G_{B(x, r_0)}^b(x, v) J^b(v, z) dv \\
&= 2c_1 r_0^{-1+\alpha-\beta} K_{B(x, r_0)}^b(x, z).
\end{aligned}$$

Here $c_1 = c_1(d, \alpha, \beta, M_1, M_2) > 0$. Thus we can continue the estimate in (5.4) to get

$$\begin{aligned}
& \partial_{x_d} g_{b,r}(x) \\
&\geq \int_{\overline{B(x, r_0)}^c} g_{b,r}(z) \left(\int_{B(x, r_0)} \partial_{x_d} G_{B(x, r_0)}(x, y) J^b(y, z) dy \right) dz \\
&\quad - \frac{2c_1}{r_0} r_0^{\alpha-\beta} \int_{\overline{B(x, r_0)}^c} g_{b,r}(z) K_{B(x, r_0)}^b(x, z) dz \\
&= \int_{\overline{B(x, r_0)}^c} g_{b,r}(z) \left(\int_{B(x, r_0)} \partial_{x_d} G_{B(x, r_0)}(x, y) J^b(y, z) dy \right) dz - \frac{2c_1}{r_0} r_0^{\alpha-\beta} g_{b,r}(x). \quad (5.5)
\end{aligned}$$

Note that by (2.8),

$$\begin{aligned}
& \partial_{x_d} G_{B(x, r_0)}(x, y) \\
&= 2^{1-\alpha} \pi^{-d/2} \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)^{-2} \frac{y_d - x_d}{|y - x|^2} r_0^\alpha (r_0^2 - |y - x|^2)^{\alpha/2} (r_0^2(r_0^2 - |y - x|^2) + |y - x|^2)^{-d/2} \\
&\quad + (d - \alpha) \frac{y_d - x_d}{|y - x|^2} G_{B(x, r_0)}(x, y). \quad (5.6)
\end{aligned}$$

Obviously $\partial_{x_d} G_{B(x, r_0)}(x, y)$ is anti-symmetric in y with respect to the hyperplane $\mathcal{H} := \{y \in \mathbb{R}^d : y_d = x_d\}$. For every $z \in \overline{B(x, r_0)}^c$, define $h_x(z) := \int_{B(x, r_0)} \partial_{x_d} G_{B(x, r_0)}(x, y) J^b(y, z) dy$. By (5.3) we have for a.e. $z \in \overline{B(x, r_0)}^c$,

$$\begin{aligned}
h_x(z) &= \int_{\{y \in B(x, r_0), y_d > x_d\}} \partial_{x_d} G_{B(x, r_0)}(x, y) \left(J^b(y, z) - J^b(\bar{y}, z) \right) dy \\
&= \int_{\{y \in B(x, r_0), y_d > x_d\}} \partial_{x_d} G_{B(x, r_0)}(x, y) \left(j^b(\tilde{y}, |z - y|) - j^b(\tilde{y}, |z - \bar{y}|) \right) dy, \quad (5.7)
\end{aligned}$$

where $\bar{y} = (\tilde{y}, 2x_d - y_d)$. We observe that the right hand side of (5.7) is antisymmetric in z with respect to the hyperplane \mathcal{H} . Recall that $j^b(\tilde{y}, r)$ is non-increasing in r . Following from this, the monotonicity of $g_{b,r}(x)$ in x_d and the Harnack inequality, we have

$$\int_{\overline{B(x, r_0)}^c} g_{b,r}(z) h_x(z) dz \geq \int_{B(\hat{x}, r_0) \cup B(\tilde{x}, r_0)} g_{b,r}(z) h_x(z) dz = \int_{B(\hat{x}, r_0)} g_{b,r}(z) h_x(z) dz$$

$$\geq c_2 g_{b,r}(x) \int_{B(\hat{x}, r_0)} h_x(z) dz \quad (5.8)$$

for some $c_2 = c_2(d, \alpha, \beta, \lambda_0, M_1, M_2) > 0$. Note that by (5.6) and (5.7), for a.e. $z \in B(\hat{x}, r_0)$,

$$\begin{aligned} h_x(z) &\geq (d - \alpha) \int_{\{y \in B(x, r_0) : y_d > x_d\}} \frac{y_d - x_d}{|y - x|^2} G_{B(x, r_0)}(x, y) \left(j^b(\tilde{y}, |z - y|) - j^b(\tilde{y}, |z - \hat{y}|) \right) dy \\ &= (d - \alpha) \int_{B(x, r_0)} \frac{y_d - x_d}{|y - x|^2} G_{B(x, r_0)}(x, y) j^b(\tilde{y}, |z - y|) dy > 0. \end{aligned}$$

Therefore by the scaling property of the Green function $G_{B(x, r_0)}$,

$$\begin{aligned} &\int_{B(\hat{x}, r_0)} h_x(z) dz \\ &\geq (d - \alpha) \int_{B(\hat{x}, r_0)} \int_{B(x, r_0)} \frac{y_d - x_d}{|y - x|^2} G_{B(x, r_0)}(x, y) J^b(y, z) dy dz \\ &\geq \frac{d - \alpha}{M_2} \int_{|v - \frac{\hat{x}}{r_0}| < 1} \int_{|w - \frac{x}{r_0}| < 1} \frac{1}{r_0} \frac{w_d - x_d/r_0}{|w - x/r_0|^2} G_{B(0, r_0)}(0, r_0(w - x/r_0)) J(r_0|v - w|) r_0^{2d} dw dv \\ &= \frac{d - \alpha}{r_0 M_2} \int_{|v - \frac{\hat{x}}{r_0}| < 1} \int_{|w - \frac{x}{r_0}| < 1} \frac{w_d - x_d/r_0}{|w - x/r_0|^2} G_{B(0, 1)}(0, w - x/r_0) J(|v - w|) dw dv \\ &= \frac{d - \alpha}{r_0 M_2} \int_{|v| < 1} \int_{|w| < 1} \frac{w_d}{|w|^2} G_{B(0, 1)}(0, w) J(|v - w + \frac{\hat{x} - x}{r_0}|) dw dv \\ &= \frac{d - \alpha}{r_0 M_2} \int_{|v| < 1} \int_{|w| < 1} \frac{w_d}{|w|^2} G_{B(0, 1)}(0, w) J(|v - w + 4\sqrt{1 + \lambda_0^2 e_d}|) dw dv =: \frac{c_3}{r_0}, \end{aligned} \quad (5.9)$$

with $c_3 = c_3(d, \alpha, M_2) > 0$. It follows from (5.5), (5.6), (5.8) and (5.9) that

$$\partial_{x_d} g_{b,r}(x) \geq \frac{1}{r_0} \left(c_2 c_3 - 2c_1 r_0^{\alpha - \beta} \right) g_{b,r}(x) \geq \frac{2}{\delta_D(x)} \left(c_2 c_3 - 2c_1 (r_2/4)^{\alpha - \beta} \right) g_{b,r}(x). \quad (5.10)$$

The lemma now follows from (5.10) by setting r_2 so small that $2c_1 (r_2/4)^{\alpha - \beta} \leq c_2 c_3/2$. \square

Lemma 5.4. *Let $r_2 \in (0, r_1] \subset (0, 1]$ be the constant in Lemma 5.3. There is a positive constant $C_{17} = C_{17}(d, \alpha, \beta, \lambda_0, M_1, M_2)$ such that for every $r \in (0, r_2]$, there is a constant $r_3 = r_3(d, \alpha, \beta, \lambda_0, M_1, M_2, r) \in (0, r/2)$ so that for every $z_0 \in \partial D$ and every non-negative function f that is regular harmonic in $D \cap B(z_0, 2r)$ with respect to X^b and vanishes in $D^c \cap B(z_0, 2r)$,*

$$|\partial_{x_d} f(x)| \geq C_{17} \frac{f(x)}{\delta_D(x)} \quad \text{for } x \in D \cap B(z_0, r_3).$$

Proof. Without loss of generality we assume $z_0 = 0$. For $r \in (0, r_2]$, fix an arbitrary $x \in D \cap B(0, r/(2\sqrt{1 + \lambda_0^2}))$. Let $z_x \in \partial D$ be such that $|x - z_x| = \rho(x)$. Define $c := \lim_{D \ni y \rightarrow z_x} f(y)/g_{b,r}(y)$ and $u(y) := c g_{b,r}(y)$. Obviously $B(z_x, 3r/2) \subset B(0, 2r)$, and thus f, u are harmonic in $D \cap B(z_x, 3r/2)$ and vanish in $D^c \cap B(z_x, 3r/2)$. Since $\lim_{D \ni y \rightarrow z_x} u(y)/f(y) = \lim_{D \ni y \rightarrow z_x} f(y)/u(y) = 1$, by Lemma 3.7, for any $y \in D \cap B(z_x, 3r/4)$,

$$\left| \frac{u(y)}{f(y)} - 1 \right| \vee \left| \frac{f(y)}{u(y)} - 1 \right| \leq c_1 \left(\frac{|y - z_x|}{r} \right)^{\gamma_1} \leq c_2 \quad (5.11)$$

for some positive constants $c_i = c_i(d, \alpha, \beta, \lambda_0, M_1, M_2)$, $i = 1, 2$. Consequently,

$$(1 + c_2)^{-1} f(y) \leq u(y) \leq (1 + c_2) f(y) \quad \text{for } y \in D \cap B(z_x, 3r/4). \quad (5.12)$$

Note that $x \in D \cap B(z_x, 3r/4)$ since $\rho(x) \leq \delta_D(x) \sqrt{1 + \lambda_0^2} \leq r/2$. By Lemma 5.3 and (5.12), we have

$$\begin{aligned} \partial_{x_d} f(x) &\geq \partial_{x_d} u(x) - |\partial_{x_d}(f - u)(x)| \geq c_3 \frac{u(x)}{\delta_D(x)} - |\partial_{x_d}(f - u)(x)| \\ &\geq c_4 \frac{f(x)}{\delta_D(x)} - |\partial_{x_d}(f - u)(x)|. \end{aligned} \quad (5.13)$$

We assume $\rho(x) < 3r/128$. Set $v(y) := f(y) - u(y)$ and $\xi := 2\rho(x)$. Let $\eta \in (16\rho(x), 3r/8)$ to be specified later. In the rest of this proof, we set $D_1 := D^+(z_x, \xi, \xi)$ and $D_2 := D^+(z_x, \eta, \eta)$. Then $B(x, \delta_D(x)) \subset D_1 \subset D_2 \subset D \cap B(z_x, 3r/4)$ and $\delta_D(x) = \delta_{D_1}(x)$. Define $V(y) := \mathbb{E}_y \left[|v|(X_{\tau_{D_1}}^b) \right]$. Clearly V is regular harmonic in D_1 with respect to X^b and $|v(y)| \leq V(y)$ for all $y \in \mathbb{R}^d$. By Theorem 1.4, we have

$$|\partial_{x_d} v(x)| \leq |\partial_{x_d} V(x)| + |\partial_{x_d}(V - v)(x)| \leq c_5 \frac{V(x)}{\delta_{D_1}(x)} = c_5 \frac{V(x)}{\delta_D(x)}. \quad (5.14)$$

We aim to estimate $V(x)$. Note that

$$\begin{aligned} V(x) &\leq \mathbb{E}_x \left[|v|(X_{\tau_{D_1}}^b) : X_{\tau_{D_1}}^b \in D_2 \right] + \mathbb{E}_x \left[f(X_{\tau_{D_1}}^b) : X_{\tau_{D_1}}^b \in D_2^c \right] + \mathbb{E}_x \left[u(X_{\tau_{D_1}}^b) : X_{\tau_{D_1}}^b \in D_2^c \right] \\ &=: I(x) + II(x) + III(x). \end{aligned}$$

By (5.11), for any $y \in D_2 \subset D \cap B(z_x, 2\eta) \subset D \cap B(z_x, 3r/4)$, we have

$$|v(y)| = f(y) \left| \frac{u(y)}{f(y)} - 1 \right| \leq c_1 f(y) \left(\frac{|y - z_x|}{r} \right)^{\gamma_1} \leq c_6 \left(\frac{\eta}{r} \right)^{\gamma_1} f(y).$$

Thus

$$I(x) \leq c_6 \left(\frac{\eta}{r} \right)^{\gamma_1} \mathbb{E}_x \left[f(X_{\tau_{D_1}}^b) \right] = c_6 \left(\frac{\eta}{r} \right)^{\gamma_1} f(x). \quad (5.15)$$

Let $A_x \in D$ be such that $\rho(A_x) = |A_x - z_x| = \eta/16$. Define $D_3 := B(A_x, \eta/16 \sqrt{1 + \lambda_0^2})$. We observe that $D_3 \subset D \cap B(z_x, \eta/2) \subset D_2$ and $D_1 \subset D \cap B(z_x, \eta/4)$. For any $y \in D_2^c \cap \text{supp} f$ and $z \in D_1$, we have $|y - A_x| \geq 7\eta/16$, $|A_x - z| \leq 5\eta/16$, and

$$|y - z| \geq |y - A_x| - |A_x - z| \geq \frac{2}{7} |y - A_x|. \quad (5.16)$$

If we let $\lambda := 1/\text{diam}(D_1)$, then $\|b_\lambda\|_\infty = \text{diam}(D_1)^{\alpha-\beta} \|b\|_\infty \leq (8\rho(x))^{\alpha-\beta} M_1 \leq M_1$. Thus by (2.5) and Lemma 3.3, we have

$$G_{D_1}^b(x, z) = \lambda^{d-\alpha} G_{\lambda D}^{b_\lambda}(\lambda x, \lambda z) \leq c_7 |x - z|^{\alpha-d}, \quad z \in D_1 \quad (5.17)$$

for some constant $c_7 = c_7(d, \alpha, \beta, M_1) > 0$. So by (5.16) and (5.17), for any $y \in D_2^c \cap \text{supp} f$,

$$K_{D_1}^b(x, y) = \int_{D_1} G_{D_1}^b(x, z) J^b(z, y) dz \leq c_7 M_2 \int_{D_1} |x - z|^{\alpha-d} J^{\varepsilon_0}(|y - z|) dz$$

$$\lesssim c_7 M_2 J^{\varepsilon_0}(|y - A_x|) \int_{B(z_x, 2\xi)} |x - z|^{\alpha-d} dz \lesssim c_7 M_2 \xi^\alpha J^{\varepsilon_0}(|y - A_x|). \quad (5.18)$$

On the other hand for any $y \in D_2^c \cap \text{supp} f$ and $z \in D_3$, we have $|y - z| \leq |y - A_x| + |A_x - z| \leq 12|y - A_x|/7$. Thus by Lemma 3.2 and (1.6)

$$\begin{aligned} K_{D_3}^b(A_x, y) &= \int_{D_3} G_{D_3}^b(A_x, z) J^b(z, y) dz \\ &\gtrsim M_2^{-1} \left(\int_{D_3} G_{D_3}(A_x, z) dz \right) J^{\varepsilon_0}(|y - A_x|) \\ &\asymp M_2^{-1} \eta^\alpha J^{\varepsilon_0}(|y - A_x|). \end{aligned} \quad (5.19)$$

Combining (5.18) and (5.19), we have

$$K_{D_1}^b(x, y) \lesssim \frac{\xi^\alpha}{\eta^\alpha} K_{D_3}^b(A_x, y) \quad \text{for } y \in D_2^c \cap \text{supp} f. \quad (5.20)$$

Consequently, by (5.20), Lemma 5.1 and (5.1), we have

$$\begin{aligned} II(x) &= \int_{D_2^c \cap \text{supp} f} f(y) K_{D_1}^b(x, y) dy \lesssim \frac{\xi^\alpha}{\eta^\alpha} \int_{D_2^c \cap \text{supp} f} f(y) K_{D_3}^b(A_x, y) dy \\ &\leq \frac{\xi^\alpha}{\eta^\alpha} \int_{D_3^c} f(y) K_{D_3}^b(A_x, y) dy = \frac{\xi^\alpha}{\eta^\alpha} f(A_x) \\ &\lesssim \frac{\xi^\alpha}{\eta^\alpha} \frac{\rho(A_x)^{\alpha-\gamma_2}}{\rho(x)^{\alpha-\gamma_2}} f(x) \asymp \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} f(x). \end{aligned} \quad (5.21)$$

Similarly we can prove that

$$III(x) \lesssim \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} u(x) \lesssim \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} f(x). \quad (5.22)$$

Combining (5.15), (5.21) and (5.22), we have

$$V(x) \leq \left(c_6 \left(\frac{\eta}{r} \right)^{\gamma_1} + c_8 \left(\frac{\rho(x)}{\eta} \right)^{\gamma_2} \right) f(x). \quad (5.23)$$

Thus by (5.13), (5.14) and (5.23) we have

$$\partial_{x_d} f(x) \geq \left(c_4 - c_5 \left(c_6 \frac{\eta^{\gamma_1}}{r^{\gamma_1}} + c_8 \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} \right) \right) \frac{f(x)}{\delta_D(x)}. \quad (5.24)$$

Let $\eta = 16\rho(x)^{\gamma_2/(\gamma_1+\gamma_2)}$. The lemma now follows from (5.24) and (5.1) provided we choose r_3 small enough such that $c_5 (c_6 16^{\gamma_1} r^{-\gamma_1} + c_8 16^{-\gamma_2}) (r_3(1+\lambda_0))^{\gamma_1 \gamma_2/(\gamma_1+\gamma_2)} \leq c_4/2$. \square

Proof of Theorem 1.5: The upper bound in (1.8) was established more generally in Theorem 1.4. The lower bound follows from Lemma 5.4 and the inequality $|\nabla f| \geq |\partial_{x_d} f|$. \square

Remark 5.5. Taking $b(x, z) = \varepsilon \in (0, M_2]$ in Assumption 1, we get from Theorem 1.4 and Theorem 1.7 the uniform gradient estimate for mixed stable processes, which in particular recovers a main result of [22] on gradient estimates.

6 Examples

In this section, we give some concrete examples where Assumptions 1 and 3 hold.

Example 6.1. If $b(x, z) = 1_{\{|z| \leq c_1\}}$ for some $c_1 > 0$, the jumping kernel of the corresponding Feller process X^b is

$$J^b(x, y) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta)}{|x - y|^{d+\beta}} 1_{\{|x-y| \leq c_1\}}.$$

In this case X^b is the independent sum of a symmetric α -stable process and a truncated symmetric β -stable process, and Assumptions 1 and 3 hold with $\varepsilon_0 = 0$ and $\psi(r) = 1_{\{r \leq c_1\}}$, respectively.

More generally, suppose $b(x, z) = b_1(x, z)1_{\{|z| \leq c_1\}}$ for some $c_1 > 0$ and a bounded function $b_1(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ that is symmetric in z and is bounded between two positive constants. Then Assumption 1 holds with $\varepsilon_0 = 0$. \square

Example 6.2. If $b(x, z) = 1 + \frac{\mathcal{A}(d, -\gamma)}{\mathcal{A}(d, -\beta)} |z|^{\beta-\gamma} 1_{\{|z| \leq c_2\}}$ for some $c_2 > 0$ and $0 < \gamma < \beta$, the jumping kernel of the corresponding Feller process X^b is

$$J^b(x, y) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta)}{|x - y|^{d+\beta}} + \frac{\mathcal{A}(d, -\gamma)}{|x - y|^{d+\gamma}} 1_{\{|x-y| \leq c_2\}}.$$

In this case X^b is the independent sum of a mixed-stable process and a truncated symmetric γ -stable process, and Assumptions 1 and 3 hold with $\varepsilon_0 = 1$ and $\psi(r) = 1 + \frac{\mathcal{A}(d, -\gamma)}{\mathcal{A}(d, -\beta)} r^{\beta-\gamma} 1_{\{r \leq c_2\}}$, respectively.

More generally, suppose $b(x, z)$ is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ that is symmetric in z and is bounded between two positive constants. Then Assumption 1 holds with $\varepsilon_0 = 1$. \square

Example 6.3. We consider the following stochastic differential equation on \mathbb{R}^d :

$$dX_t = dY_t + C(X_{t-})dZ_t, \tag{6.1}$$

where Y is a symmetric α -stable process, Z is an independent β -stable process with $0 < \beta < \alpha$, and C is a bounded Lipschitz function on \mathbb{R}^d . Using Picard's iteration method, one can show that for every $x \in \mathbb{R}^d$, SDE (6.1) has a unique strong solution with $X_0 = x$. The collection of the solutions $(X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ forms a strong Markov process X on \mathbb{R}^d . Using Ito's formula, one concludes that the infinitesimal generator of X is \mathcal{L}^b with $b(x, z) = |C(x)|^\beta$. If there exists $c_3 > 0$ such that $|C(x)| \geq c_3$ for $x \in \mathbb{R}^d$, then our Assumption 1 holds with $\varepsilon_0 = 1$. \square

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